

# FRANC3D Training Workshop: Part 8

## SIFs from FE Analysis

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# Workshop Agenda

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- Part 1: Introduction to Fatigue and Damage Tolerance
- Part 2: Introduction to Fracture Mechanics Analysis
- Part 3: Introduction to FRANC3D
- Part 4: FRANC3D User Interface
- Part 5: Finite Element (FE) Model Import
- Part 6: Crack Insertion
- Part 7: Static Crack Analysis & SIF Computation
- **Part 8: SIFs from FE Analysis**
- Part 9: Crack Growth
- Part 10: SIF History & Fatigue Life
- Part 11: Miscellaneous Topics

# SIFs from FE Analysis

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- Linear Elastic Fracture Mechanics Using FE Analysis
- Stress Intensity Factors (SIFs)
- Computing Stress Intensity Factors
  - Displacement Correlation Method
  - Conservative Integral Method
    - J-Integral
    - M-Integral
- Benchmark Examples Using M-integral
- Computing SIFs FRANC3D User Interface

# Linear Elastic Fracture Mechanics Analysis Using Finite Elements

# LEFM Using FEM

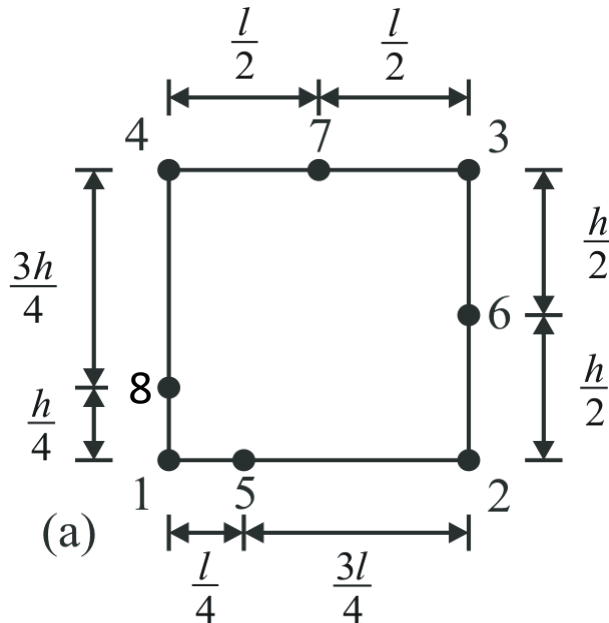
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- Finite element method is a natural tool for performing LEFM problems
- Number of researchers investigated **special finite element formulations** that incorporate **singular basis functions** or **stress intensity factors as nodal variables**
  - These special elements are **not** available in most general-purpose finite element programs
- Finite Element Method (FEM) needs to be manipulated to handle the singular crack-tip stress and strain fields predicted by theory

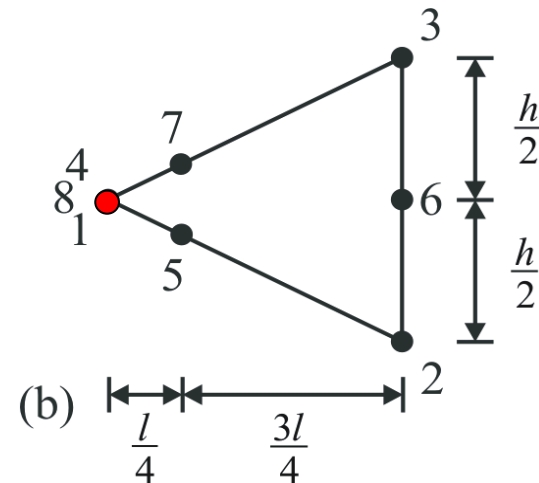
# LEFM Using FEM

- A significant advancement in the use of the finite element method for LEFM problems was the simultaneous, and independent, development of the "quarter-point element" .

Henshell and Shaw, 1975



Barsoum, 1976



Quadrilateral (a) and collapsed quadrilateral (b) quarter-point elements

# LEFM Using FEM

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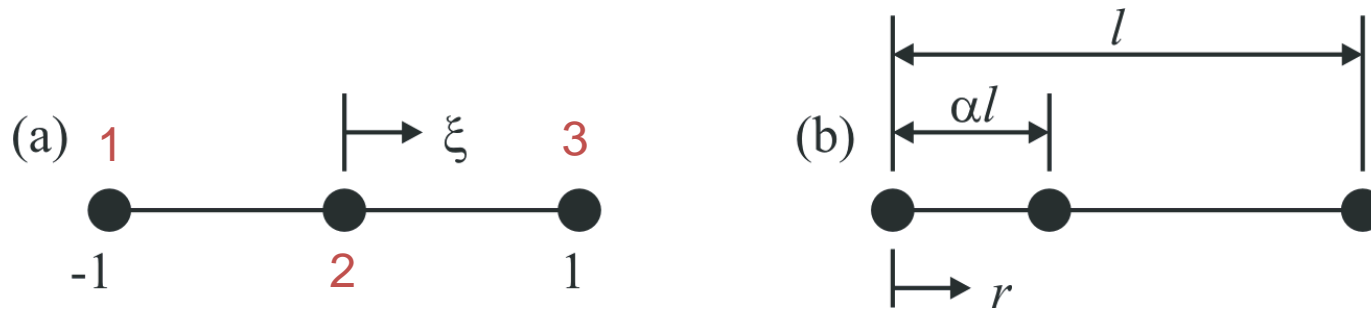
- These researchers showed that proper crack-tip displacement, stress, and strain fields are modeled by standard, quadratic order, isoparametric finite elements if one simply moves the element's mid-side node to the quarter-point position.
  - This introduces a singularity into the mapping between the element's parametric coordinate space and Cartesian space.

# LEFM Using FEM

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- The discovery of quarter-point elements was a significant milestone in the development of FE procedures for LEFM.
- With these elements, standard and widely available finite element programs can be used to model crack tip fields accurately with only **minimal preprocessing** required.

# 1D $\frac{1}{4}$ -Point Singular Element



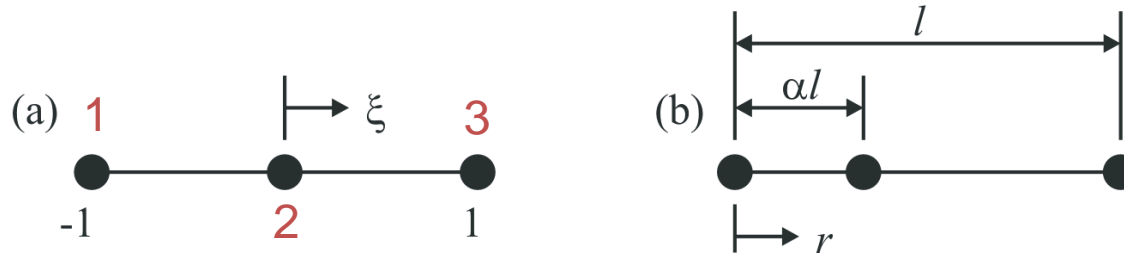
A quadratic element, where (a) shows the parametric space of the element, and (b) shows the Cartesian space of the element. The crack tip is at  $r=0$ .

$$\mathbf{u} = \sum_{i=1}^3 N_i \mathbf{u}_i = \frac{1}{2} \xi(\xi - 1) \mathbf{u}_1 + (1 - \xi^2) \mathbf{u}_2 + \frac{1}{2} \xi(\xi + 1) \mathbf{u}_3$$

The standard polynomial *displacement* interpolation scheme is regrouped:

$$\mathbf{u} = \mathbf{u}_2 + \frac{1}{2} (\mathbf{u}_3 - \mathbf{u}_1) \xi + \left( \frac{1}{2} (\mathbf{u}_1 + \mathbf{u}_3) - \mathbf{u}_2 \right) \xi^2$$

# 1D ¼-Point Singular Element



$$r = \sum_{i=1}^3 N_i r_i = \alpha l + \frac{1}{2} l \xi + l \left( \frac{1}{2} - \alpha \right) \xi^2$$

and the standard, polynomial *geometry* interpolation scheme.

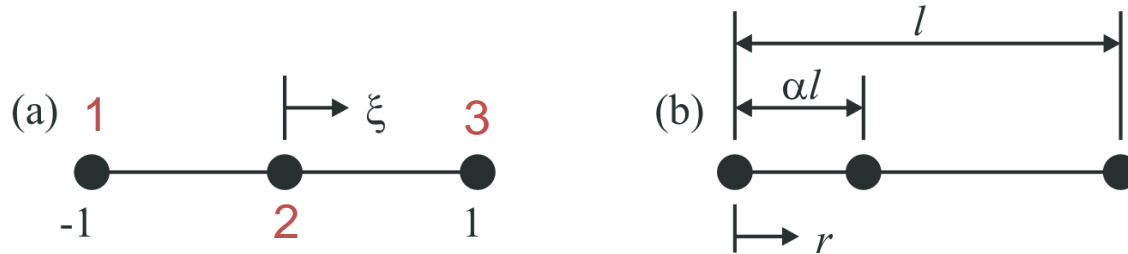
First, the usual case of mid-side geometry:

$$\alpha = \frac{1}{2} \longrightarrow \xi = \frac{2r}{l} - 1$$

The expected polynomial interpolation is:

$$\mathbf{u} = \mathbf{u}_1 + (-3\mathbf{u}_1 + 4\mathbf{u}_2 - \mathbf{u}_3) \frac{r}{l} + 2(\mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3) \frac{r^2}{l^2}$$

# 1D 1/4-Point Singular Element



$$r = \sum_{i=1}^3 N_i r_i = \alpha l + \frac{1}{2} l \xi + l \left( \frac{1}{2} - \alpha \right) \xi^2$$

The standard polynomial geometry interpolation scheme.

Next, the unusual case of 1/4-point geometry:

$$\alpha = \frac{1}{4} \longrightarrow \xi = \frac{2\sqrt{lr}}{l} - 1$$

The *unexpected* non-polynomial interpolation is:

$$\mathbf{u} = \mathbf{u}_1 + 2(\mathbf{u}_1 - 2\mathbf{u}_2 + \mathbf{u}_3) \frac{r}{l} + (-3\mathbf{u}_1 + 4\mathbf{u}_2 + \mathbf{u}_3) \frac{\sqrt{lr}}{l}$$

# 1D ¼-Point Singular Element

Normal displacement field: 
$$u = u_1 + (-3u_1 + 4u_2 - u_3) \frac{r}{l} + 2(u_1 - 2u_2 + u_3) \frac{r^2}{l^2}$$

¼-Point displacement field: 
$$u = u_1 + 2(u_1 - 2u_2 + u_3) \frac{r}{l} + (-3u_1 + 4u_2 + u_3) \frac{\sqrt{lr}}{l}$$

Compare to theoretical field: 
$$u = \sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^1 \left[ \left( \kappa + \frac{n}{2} + (-1)^n \right) \cos \frac{n}{2} \theta - \frac{n}{2} \cos \left( \frac{n}{2} - 2 \right) \theta \right]$$

Normal strain field: 
$$\varepsilon = \frac{du}{dr} = (-3u_1 + 4u_2 - u_3) \frac{1}{l} + 4(u_1 - 2u_2 + u_3) \frac{r}{l^2}$$

¼-point strain field: 
$$\varepsilon = \frac{du}{dr} = 2(u_1 - 2u_2 + u_3) \frac{1}{l} + \left( -\frac{3}{2}u_1 + 2u_2 - \frac{1}{2}u_3 \right) \frac{1}{\sqrt{lr}}$$

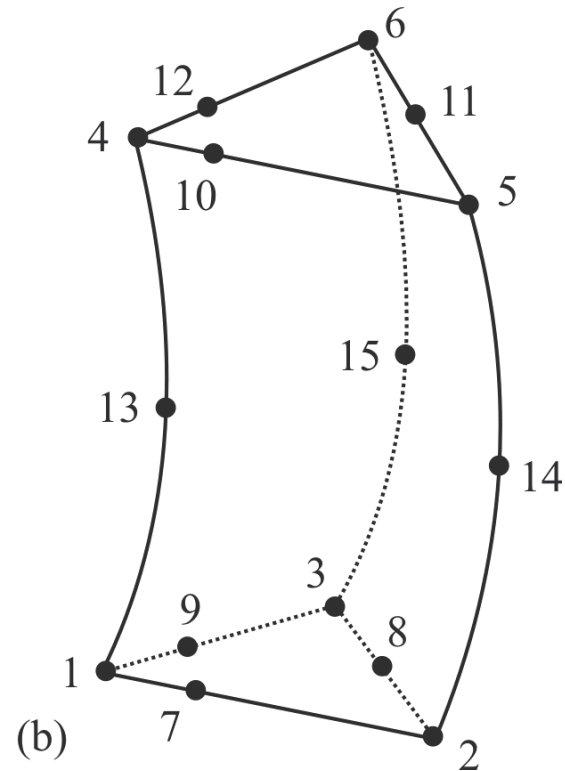
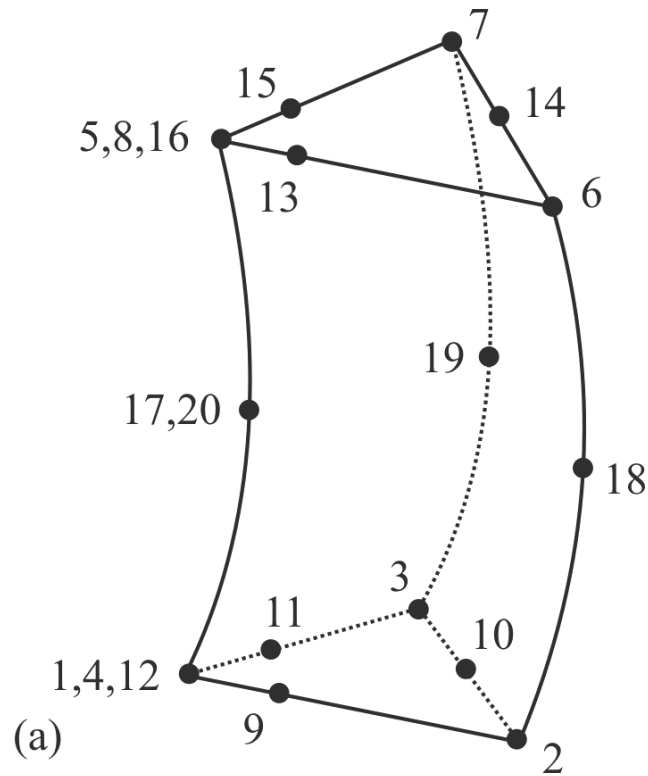
Compare to theoretical field: 
$$\sigma_x = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^1 \left[ \left( 2 + \frac{n}{2} + (-1)^n \right) \cos \left( \frac{n}{2} - 1 \right) \theta - \left( \frac{n}{2} - 1 \right) \cos \left( \frac{n}{2} - 3 \right) \theta \right]$$

# 1D $\frac{1}{4}$ -Point Singular Element

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- Displacement field:
  - Contains a constant value, a linear variation in  $r$ , and the square root variation in  $r$ .
  - Corresponds to the leading terms in the LEFM expressions for the near crack-tip displacement.
- Strain field:
  - Contains a constant term and a singular term that varies as  $r^{-1/2}$
  - Corresponds to the leading term in the LEFM stress and strain expansions

# 3D $\frac{1}{4}$ -Point Singular Elements



- (a) collapsed, 20-noded brick, quarter-point element
- (b) natural 15-noded, wedge quarter-point element

# Stress Intensity Factors

# Stress Intensity Factors

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- SIF's are a measure of the intensity of the stress concentration along a crack front
- Range in the SIF ( $\Delta K$ ) due to cyclic loading can be correlated with the crack growth rate
- SIF's are functions of the crack size and shape, local geometry of the component, and local stresses in the crack region

# Stress Intensity Factors

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- FRANC3D was developed primarily to calculate highly **accurate** SIF's for **arbitrarily** shaped cracks in **arbitrarily** shaped components subjected **arbitrary** loading
- FRANC3D **handles real-world conditions** such as non-planar cracks, cracks with complex shapes, complex component geometries, contact between components, residual stresses, and local stress concentrations

# Stress Intensity Factors

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- FRANC3D computes stress intensity factors for all three “modes” of fracture at FE **mid-side nodes along the crack front**
- Under conditions of small-scale yielding, crack **behavior** is controlled by stress intensity factors:
  - **Stability** – will the crack tip move?
  - **Trajectory** – in what direction?
  - **Rate** – how fast?

# Computing Stress Intensity Factors

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- FRANC3D has three methods of computing stress intensity factors (SIF's or K's):
  - Displacement Correlation Method
  - Conservative Integral Method
    - *J*-Integral
    - *M*-Integral
  - Virtual Crack Closure Method

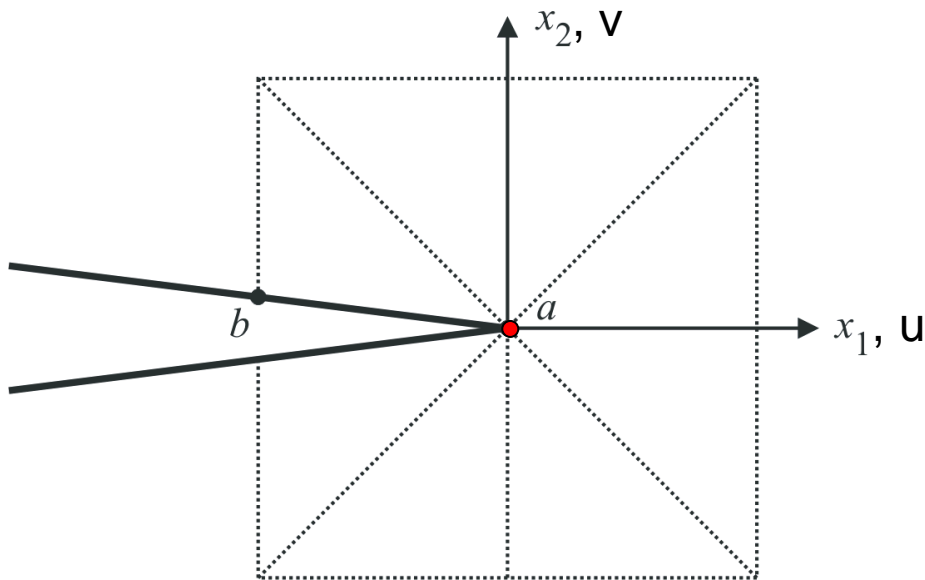
# Displacement Correlation Method

# Displacement Correlation Method

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- Relatively simple to understand and implement
- Relatively poor accuracy (~5% error for a reasonable mesh)
- Good sanity check but not for production work

# Theoretical asymptotic displacement fields



Set  $r = r_{a-b}$ , and  $\theta = 180^\circ$

$$v_b - v_a = \frac{K_I}{\mu} \left[ \frac{r_{a-b}}{2\pi} \right] (2 - 2\nu)$$

$$u = \frac{K_I}{\mu} \left[ \frac{r}{2\pi} \right]^{1/2} \cos \frac{\theta}{2} \left[ 1 - 2\nu + \sin^2 \frac{\theta}{2} \right]$$

$$v = \frac{K_I}{\mu} \left[ \frac{r}{2\pi} \right]^{1/2} \sin \frac{\theta}{2} \left[ 2 - 2\nu - \cos^2 \frac{\theta}{2} \right]$$

Note: for plane stress, let  $\nu = \nu/(1 + \nu)$

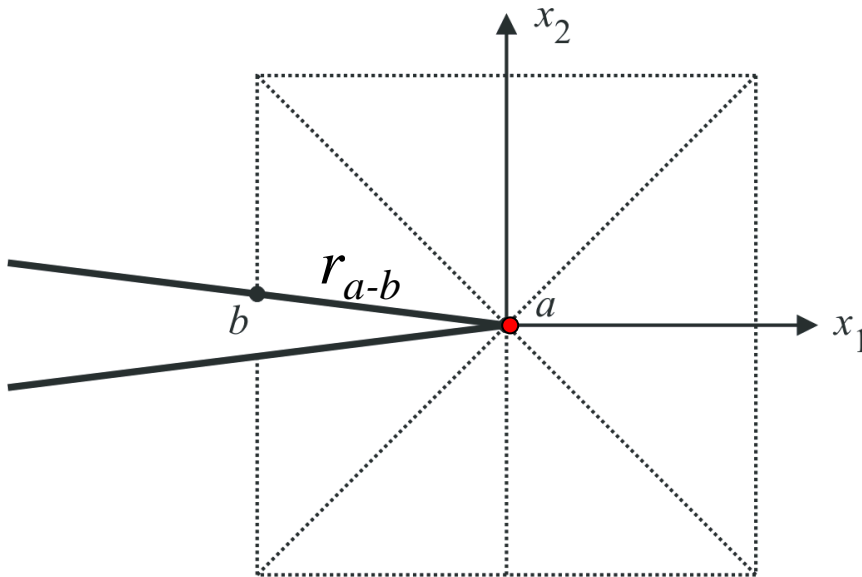
$$u = \frac{K_{II}}{\mu} \left[ \frac{r}{2\pi} \right]^{1/2} \sin \frac{\theta}{2} \left[ 2 - 2\nu + \cos^2 \frac{\theta}{2} \right]$$

$$v = \frac{K_{II}}{\mu} \left[ \frac{r}{2\pi} \right]^{1/2} \cos \frac{\theta}{2} \left[ -1 + 2\nu + \sin^2 \frac{\theta}{2} \right]$$

$$u_b - u_a = \frac{K_{II}}{\mu} \left[ \frac{r_{a-b}}{2\pi} \right] (2 - 2\nu)$$

# Displacement Correlation (No Crack-Front Template)

For plane strain case:



$$K_I = \frac{\mu\sqrt{2\pi}(v_b - v_a)}{\sqrt{r_{a-b}}(2 - 2\nu)}$$

$$K_{II} = \frac{\mu\sqrt{2\pi}(u_b - u_a)}{\sqrt{r_{a-b}}(2 - 2\nu)}$$

$$K_{III} = \frac{\mu\sqrt{\pi}(w_b - w_a)}{\sqrt{2r_{a-b}}}$$

$\mu$  is the shear modulus,  $\nu$  is Poisson's ratio,  $r$  is the distance from the crack tip to the correlation point, and  $u_i, v_i, w_i$  are the  $x, y,$  and  $z$  displacements at point  $i$

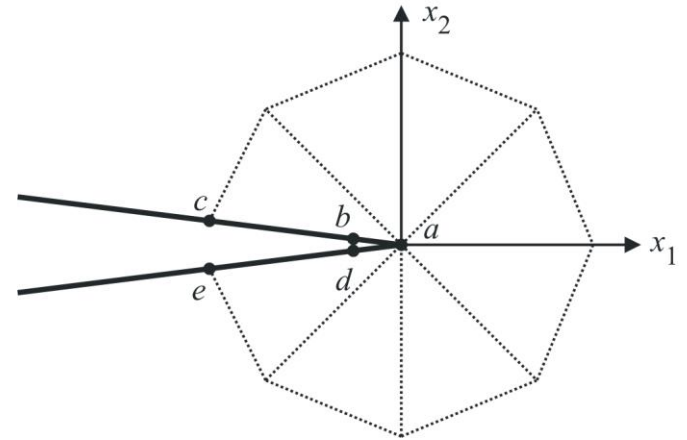
Same expressions used for plane stress assumptions if  $\nu$  is replaced by  $\nu / (1+\nu)$

# Displacement Correlation Methods

- SIF's computed by this approach can be improved if quarter-point crack-tip elements are used
- Displacement field:

$$v_{upper} = v_a + (-3v_a + 4v_b - v_c)\sqrt{\frac{r}{l}} + (2v_a - 4v_b + 2v_c)\frac{r}{l}$$

$$v_{lower} = v_a + (-3v_a + 4v_d - v_e)\sqrt{\frac{r}{l}} + (2v_a - 4v_d + 2v_e)\frac{r}{l}$$



The FE crack opening displacement (COD) is

$$v_{upper} - v_{lower} = [4(v_b - v_d) + v_e - v_c]\sqrt{\frac{r}{l}} + [4(v_b - v_d) + 2(v_c - v_e)]\frac{r}{l}$$

# Displacement Correlation (With Crack-Front Template)

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- The square root term of the FE COD can then be substituted into the analytical crack-tip displacement field to yield

$$K_I = \frac{\mu\sqrt{2\pi}}{\sqrt{r}(2-2\nu)} [4(v_b - v_d) + v_e - v_c]$$

$$K_{II} = \frac{\mu\sqrt{2\pi}}{\sqrt{r}(2-2\nu)} [4(u_b - u_d) + u_e - u_c]$$

# Conservative Integral Methods

## *M*- & *J*-Integrals

# $J$ -Integral

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- In linear Elastic Fracture Mechanics (LEFM), the  $J$ -integral is equivalent to the fracture energy release rate,  $G$ .

# Energy Release Rate, $G$

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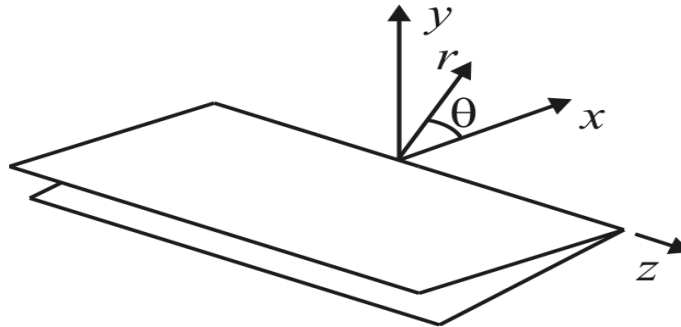
- $G$  can be computed either:
    - Using the stress intensity factors and material properties and the theoretical expressions for near crack-front displacements and stresses
- OR*
- Using the near-crack-front stresses, strains and displacement determined from a finite element analysis (domain integration)

# G Computation Using the Stress Intensity Factors and Material Properties

# Energy Release Rate Expression Using SIFs & Material Properties

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Given the following crack-front coordinate systems (Rectangular and Polar coordinate systems)



The crack-tip energy release rates can be determined from Irwin's\* crack closure integral

$$G = \lim_{D \rightarrow 0} \frac{1}{D} \int_0^D \sigma_{yi}(D-r, 0) \cdot u_i(r, \rho) dr, \quad i = x, y, z \quad (1)$$

where  $\sigma(r, \theta)$  are the near crack-front stresses (in cylindrical coordinates), and  $u(r, \theta)$  are the near crack-front displacements

\* Irwin, G.R. (1957) Analysis of stresses and strains near the end of a crack traversing a plate, *Journal of Applied Mechanics*, 24, 361-364

# Energy Release Rate Expression Using SIFs & Material Properties

The first term of Williams\* expansion for the stresses and displacements near the crack tip is related to the stress intensity factors and gives the following expression for stress and displacement assuming plane strain conditions.

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix} = \frac{1}{\sqrt{2\rho r}} \begin{bmatrix} \cos\frac{\alpha}{2}\left(1 - \sin\frac{\alpha}{2}\sin\frac{3\alpha}{2}\right) & \sin\frac{\alpha}{2}\left(2 + \cos\frac{\alpha}{2}\cos\frac{3\alpha}{2}\right) & 0 \\ \cos\frac{\alpha}{2}\left(1 + \sin\frac{\alpha}{2}\sin\frac{3\alpha}{2}\right) & \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos\frac{3\alpha}{2} & 0 \\ \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\cos\frac{3\alpha}{2} & \cos\frac{\alpha}{2}\left(1 - \sin\frac{\alpha}{2}\sin\frac{3\alpha}{2}\right) & 0 \\ 0 & 0 & -\sin\frac{\alpha}{2} \\ 0 & 0 & \cos\frac{\alpha}{2} \end{bmatrix} \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix} \quad (2)$$

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \frac{2(1+\nu)}{E} \sqrt{\frac{r}{2\rho}} \begin{bmatrix} \cos\frac{\alpha}{2}\left(1 - 2\nu + \sin^2\frac{\alpha}{2}\right) & \sin\frac{\alpha}{2}\left(2 - 2\nu + \cos^2\frac{\alpha}{2}\right) & 0 \\ \sin\frac{\alpha}{2}\left(2 - 2\nu - \cos^2\frac{\alpha}{2}\right) & \cos\frac{\alpha}{2}\left(2\nu - 1 + \sin^2\frac{\alpha}{2}\right) & 0 \\ 0 & 0 & 2\sin\frac{\alpha}{2} \end{bmatrix} \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix} \quad (3)$$

\* Williams, M.L. (1957) On the stress distribution at the base of a stationary crack, *Journal of Applied Mechanics*, 24, 109-114

# Energy Release Rate Expression Using SIFs & Material Properties

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Substituting  $\theta = 0$  into equation 2, and  $\theta = \pi$  in equation 3 gives

$$\left\{ \begin{array}{c} S_{yy} \\ S_{xy} \\ S_{yz} \end{array} \right\} = \frac{1}{\sqrt{2\rho r}} \left\{ \begin{array}{c} K_I \\ K_{II} \\ K_{III} \end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c} u_x \\ u_y \\ u_z \end{array} \right\} = \frac{2(1+n)}{E} \sqrt{\frac{r}{2\rho}} \left\{ \begin{array}{c} 2(1-n)K_I \\ 2(1-n)K_{II} \\ K_{III} \end{array} \right\} \quad (4)$$

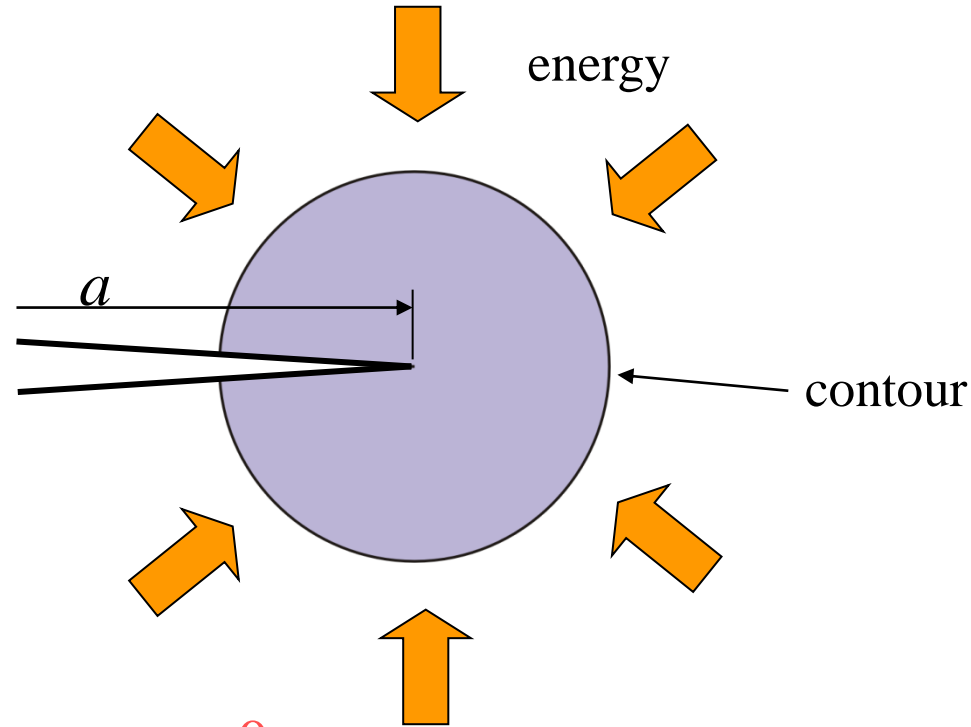
Substituting equations 4 into 1, and evaluating the integral gives the well know result

$$G = \frac{1-n^2}{E} K_I^2 + \frac{1-n^2}{E} K_{II}^2 + \frac{1+n}{E} K_{III}^2 \quad (5)$$

the energy release rate in terms of stress intensity factors and material properties

G Computation Using the near-crack-front stresses, strains and displacement determined from a finite element analysis (domain integration)

# 2D Crack-Tip Energy Balance (J-integral)



informally:

$$\frac{dE_{contour}}{da} = \frac{dE_{elastic}}{da} + \frac{dE_{plastic}}{da} + \frac{dE_{new\ surface}}{da} + \frac{dE_{heat}}{da} + \frac{dE_{sound}}{da} + \frac{dE_{light}}{da}$$

0
small

which reduces to:

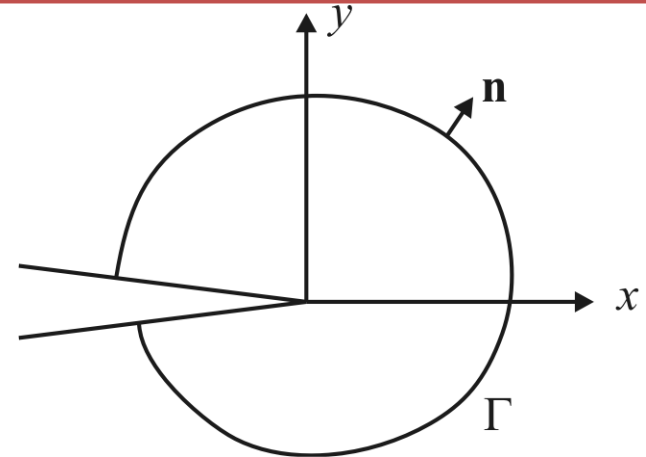
$$-\frac{dE_{new\ surface}}{da} = G = \frac{dE_{elastic}}{da} - \frac{dE_{contour}}{da} = J$$

# Energy Release Rate Expressed Using Near-Crack-front Stresses, Stains and Displacements

The  $J$ -Integral\* measures energy flux into the crack-tip region

$$J = \int_{\Gamma} \left( W n_x - T_i \frac{\partial u_i}{\partial x} \right) ds \quad W = \frac{1}{2} \sigma_{ij} \varepsilon_{ij}$$

Under conditions of small scale yielding,  **$J$ -Integral is equal to energy release rate,  $G$**



The contour  $J$ -Integral can be recast as an equivalent area (volume in 3D) integral\*\*, which is more **accurate and stable** in a finite element context

$$J' = \int_V \left( \sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right) \frac{\partial q}{\partial x_j} dv \quad (6)$$

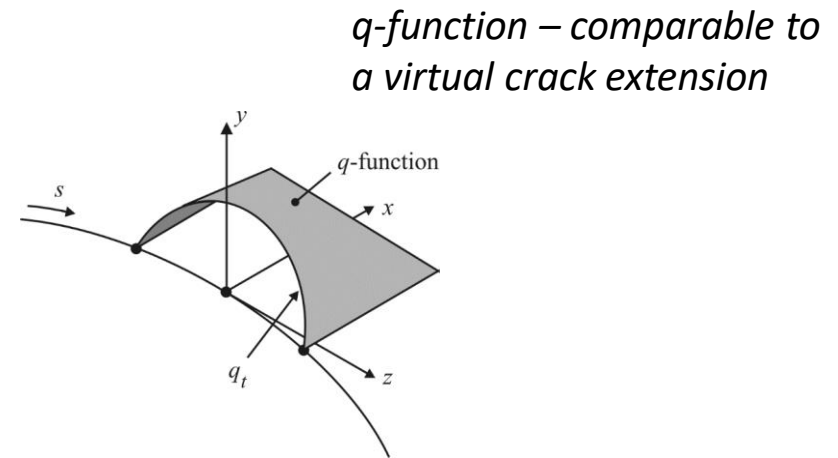
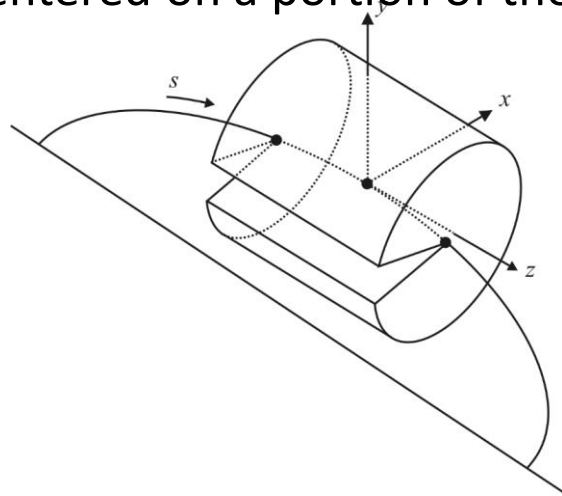
$q$  is a function that is one at the crack tip and zero on the boundary of the integration domain. It can be interpreted as a virtual crack extension, and  $\delta$  is the Kronecker delta

\* Rice, J.R. (1968) A path independent integral and approximate analysis of strain concentrations by notches and cracks, *Journal of Applied Mechanics*, 35, 379-386

\*\* Li, F.Z., Shih, C.F., and Needleman, A. (1985) A comparison of methods for calculating energy release rates, *Engineering Fracture Mechanics*, 21, 405-421

# Energy Release Rate Expressed Using Near-Crack-front Stresses, Strains and Displacements

In 3D, the J-Integral is evaluated within a cylindrical domain centered on a portion of the crack-front.



$J'$  is the energy release for the specified virtual crack extension  $q$ . This must be normalized to get the energy released for a unit amount of new crack area

$$G = \frac{J'}{\int q(s) ds} = \frac{J'}{A_q} = \int_V \left( S_{ij} \frac{\partial u_i}{\partial x_j} - W d_{1j} \right) \frac{\partial q}{\partial x_j} dV \Big/ A_q \quad (7)$$

This is an expression for the energy release rate in terms of stresses, strains, and displacements

# $J$ -Integral

- The problem with  $J$  is that it only gives **one number** for the fracture energy release rate
  - Difficult to segregate the fracture energy release rate among the three modes of fracture.

# *M*-Integral (also called the interaction integral)

# Computing Stress Intensity Factors

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## M-Integral (Interaction Integral):

- Numerically,  $M$ -Integral is similar to  $J$ -Integral
- $M$ -Integral computes strain energy release rates ( $G_I, G_{II}, G_{III}$ ) and stress intensity factors ( $K_I, K_{II}, K_{III}$ ) associated with three modes of fracture
  - Mode II ( $K_{II}$ ) needed to predict kink angle for crack growth direction
- $M$ -integral implementation in FRANC3D works for isotropic and anisotropic materials
- Relatively good accuracy (<1% error for a [reasonable mesh density...no need for super-refined mesh at the crack front](#))
- Requires additional terms for crack face tractions, residual stresses, thermal strains, etc.

<sup>1</sup> Wawrzynek, P.A., Carter, B., and Banks-Sills, L. "The  $M$ -integral for computing stress intensity factors in generally anisotropic materials," NASA/CR-2005-214006. July 2005

<sup>2</sup> Banks-Sills, L., Wawrzynek, P.A., Carter, B., Ingraffea, T.R., and Hershkovitz, I., "Methods for computing stress intensity factors in anisotropic geometries: Part II – arbitrary geometry," *Engng. Fracture Mech*, Vol 74, No 8, pp 1293-1307. July 2006

# Steps for Formulating the M-Integral

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- **Step(1):** Invoke superposition to combine the finite element solution with a known analytical solution
- **Step(2):** Substitute the superposition terms into the energy release rate ( $G$ ) equations (5) and (7) from the previous slides
- **Step(3):** Equate the two expressions for  $G$  and substitute a simple analytical  $K_I$  solution, and solve for the finite element  $K_I$
- **Step (4):** Repeat step (3) for  $K_{II}$  and  $K_{III}$

# Principal of Superposition

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The principal of superposition for elasticity says that if you have one solution that satisfies the governing equations for elasticity, and you add it to a second solution that satisfies these equations, the result will also satisfy the governing equations

Therefore,

$$K_I = K_I^{(1)} + K_I^{(2)}, \quad K_{II} = K_{II}^{(1)} + K_{II}^{(2)}, \quad K_{III} = K_{III}^{(1)} + K_{III}^{(2)} \quad (8)$$

$$S_{ij} = S_{ij}^{(1)} + S_{ij}^{(2)}, \quad e_{ij} = e_{ij}^{(1)} + e_{ij}^{(2)}, \quad u_i = u_i^{(1)} + u_i^{(2)} \quad (9)$$

where the (1) solution is the finite element results  
and (2) is an analytical solution

# Substitution in The Energy Release Rate Equations

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Substituting equation (8) into equation (5) (*energy release rate in terms of stress intensity factors and material properties*), and collecting terms gives

$$G = G^{(1)} + G^{(2)} + M^{(1,2)} \quad (10)$$

where

$$M^{(1,2)} = \frac{1 - \eta^2}{E} K_I^{(1)} K_I^{(2)} + \frac{1 - \eta^2}{E} K_{II}^{(1)} K_{II}^{(2)} + \frac{1 + \eta}{E} K_{III}^{(1)} K_{III}^{(2)} \quad (11)$$

# Substitution in The Energy Release Rate Equations

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Substituting equation (9) into equation (7) (*energy release rate in terms of the near-crack-front stresses, strains and displacements*), and collecting terms gives

$$G = G^{(1)} + G^{(2)} + M^{(1,2)} \quad (12)$$

Where in this case

$$M^{(1,2)} = \int_V \left( S_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + S_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} d_{1j} \right) \frac{\partial q}{\partial x_j} dV \Big/ A_q \quad (13)$$

# Solving for The Finite Element $K_I^{(1)}$

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For the  $(2)$  solution, we use the first term of the Williams expansion, (equations 2 and 3) after setting  $K_I = 1$  and  $K_{II} = K_{III} = 0$ . Substituting this into equation 11 and 13, and setting the two expressions for  $M$  equal, gives

$$\frac{1 - \nu^2}{E} K_I^{(1)} = \int_V \left( S_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + S_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} d_{1j} \right) \frac{\partial q}{\partial x_j} dV \Big/ A_q \quad (14)$$

or, the finite element  $K_I$  is

$$K_I^{(1)} = \frac{E}{1 - \nu^2} \int_V \left( S_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + S_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} d_{1j} \right) \frac{\partial q}{\partial x_j} dV \Big/ A_q \quad (15)$$

where  $\sigma^{(1)}$  and  $u^{(1)}$  come from finite element results, and  $\sigma^{(2)}$  and  $u^{(2)}$  are (next slide)

# Solving for The Finite Element $K_I^{(1)}$

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix} = \frac{1}{\sqrt{2\rho r}} \begin{Bmatrix} \cos\frac{q}{2}\left(1 - \sin\frac{q}{2}\sin\frac{3q}{2}\right) \\ \cos\frac{q}{2}\left(1 + \sin\frac{q}{2}\sin\frac{3q}{2}\right) \\ n(S_{xx} + S_{yy}) \\ \sin\frac{q}{2}\cos\frac{q}{2}\cos\frac{3q}{2} \\ 0 \\ 0 \end{Bmatrix} \quad (16)$$

and

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \frac{2(1+n)}{E} \sqrt{\frac{r}{2\rho}} \begin{Bmatrix} \cos\frac{q}{2}\left(1 - 2n + \sin^2\frac{q}{2}\right) \\ \sin\frac{q}{2}\left(2 - 2n - \cos^2\frac{q}{2}\right) \\ 0 \end{Bmatrix} \quad (17)$$

These come from equations 2 and 3 with  $K_I = 1$  and  $K_{II} = K_{III} = 0$

# Solving for The Finite Element $K_{II}^{(1)}$

---

For the  $(2)$  solution, we use the first term of the Williams expansion, (equations 2 and 3) after setting  $K_I = 0$ ,  $K_{II} = 1$  and  $K_{III} = 0$ . Substituting this into equation 11 and 13, and setting the two expressions for  $M$  equal, the finite element  $K_{II}$  is

$$K_{II}^{(1)} = \frac{E}{1 - \nu^2} \int_V \left( S_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + S_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} d'_{1j} \right) \frac{\partial q}{\partial x_j} dV \bigg/ A_q \quad (18)$$

where  $\sigma^{(1)}$  and  $u^{(1)}$  come from finite element results, and  $\sigma^{(2)}$  and  $u^{(2)}$  are (next slide)

# Solving for The Finite Element $K_{II}^{(1)}$

---

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix} = \frac{1}{\sqrt{2\rho r}} \begin{Bmatrix} \sin \frac{q}{2} \left( 2 + \cos \frac{q}{2} \cos \frac{3q}{2} \right) \\ \sin \frac{q}{2} \cos \frac{q}{2} \cos \frac{3q}{2} \\ n(S_{xx} + S_{yy}) \\ \cos \frac{q}{2} \left( 1 - \sin \frac{q}{2} \sin \frac{3q}{2} \right) \\ 0 \\ 0 \end{Bmatrix} \quad (19)$$

and

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \frac{2(1+n)}{E} \sqrt{\frac{r}{2\rho}} \begin{Bmatrix} \sin \frac{q}{2} \left( 2 - 2n + \cos^2 \frac{q}{2} \right) \\ \cos \frac{q}{2} \left( -1 + 2n + \sin^2 \frac{q}{2} \right) \\ 0 \end{Bmatrix} \quad (20)$$

These come from equations 2 and 3 with  $K_I = 0$ ,  $K_{II} = 1$  and  $K_{III} = 0$

# Solving for The Finite Element $K_{III}^{(1)}$

---

For the  $(^2)$  solution, we use the first term of the Williams expansion, (equations 2 and 3) after setting  $K_I = K_{II} = 0$  and  $K_{III} = 1$ . Substituting this into equation 11 and 13, and setting the two expressions for  $M$  equal, the finite element  $K_{III}$  is

$$K_{III}^{(1)} = \frac{E}{1+n} \int_V \left( S_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + S_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} d_{1j} \right) \frac{\partial q}{\partial x_j} dV \Big/ A_q \quad (21)$$

where  $\sigma^{(1)}$  and  $u^{(1)}$  come from finite element results, and  $\sigma^{(2)}$  and  $u^{(2)}$  are (next slide)

# Solving for The Finite Element $K_{III}^{(1)}$

---

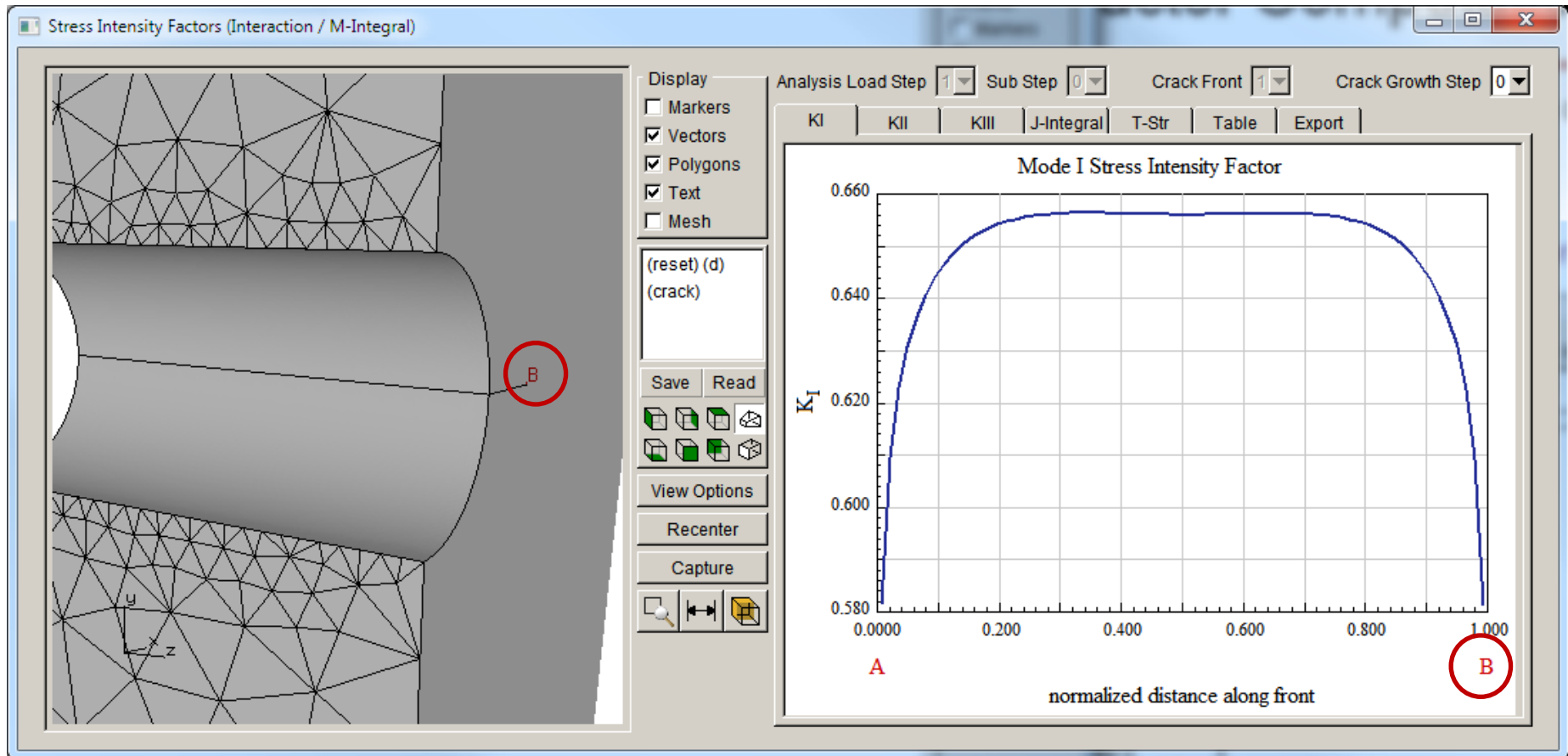
$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix} = \frac{1}{\sqrt{2\rho r}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \sin\frac{q}{2} \\ \cos\frac{q}{2} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \frac{2(1+\nu)}{E} \sqrt{\frac{r}{2\rho}} \begin{Bmatrix} 0 \\ 0 \\ \sin\frac{q}{2} \end{Bmatrix} \quad (22)$$

These come from equations 2 and 3 with  $K_I = 0$ ,  $K_{II} = 0$  and  $K_{III} = 1$

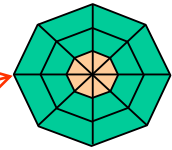
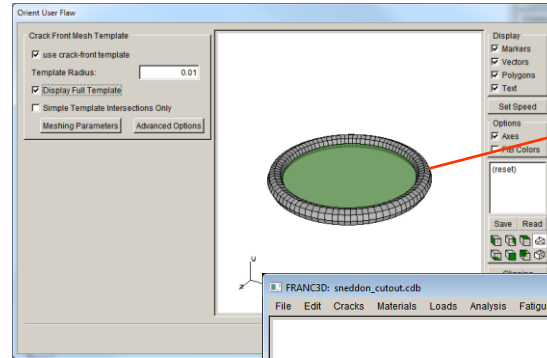
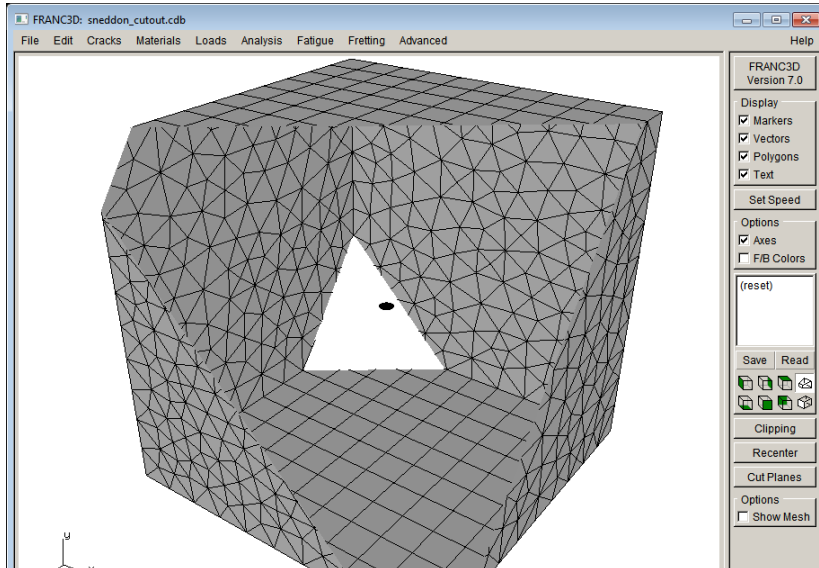
# Benchmark Examples Using M-integral

# Stress Intensity Factor Computations

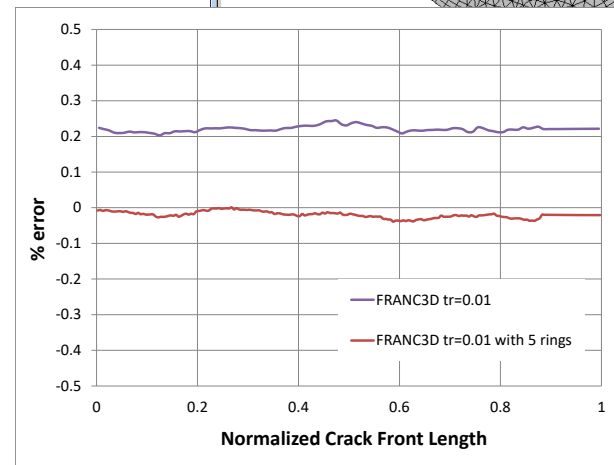
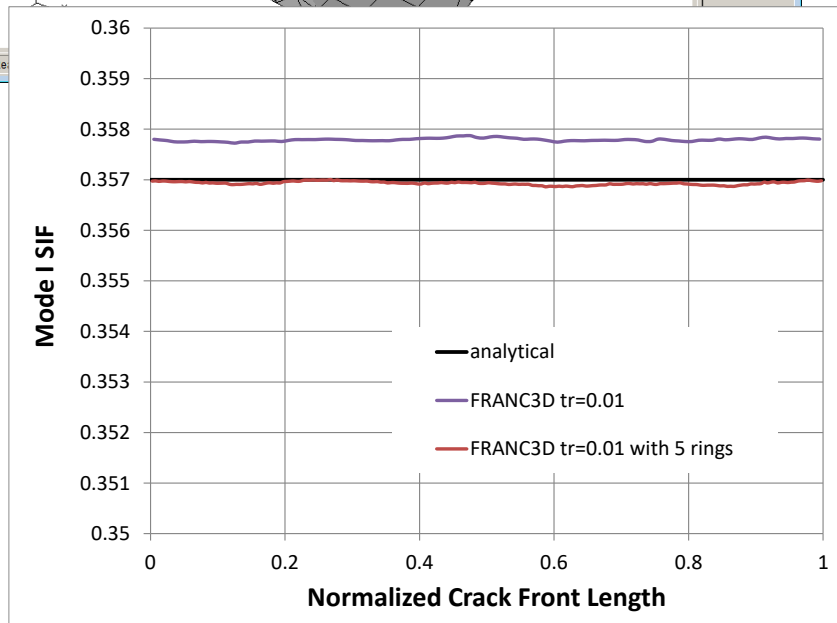
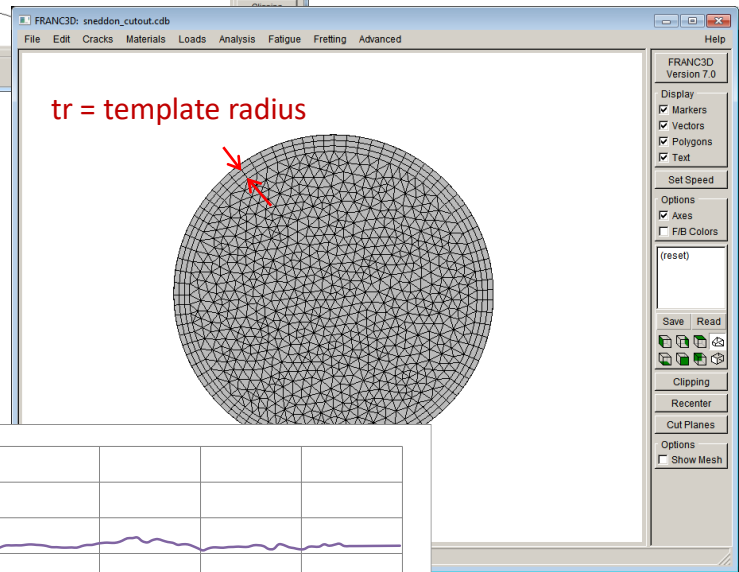
FRANC3D displays stress intensity factors along the normalized (from 0 to 1) crack front length, starting at **A** and ending at **B**



# M-Integral SIFs: Embedded Penny Crack

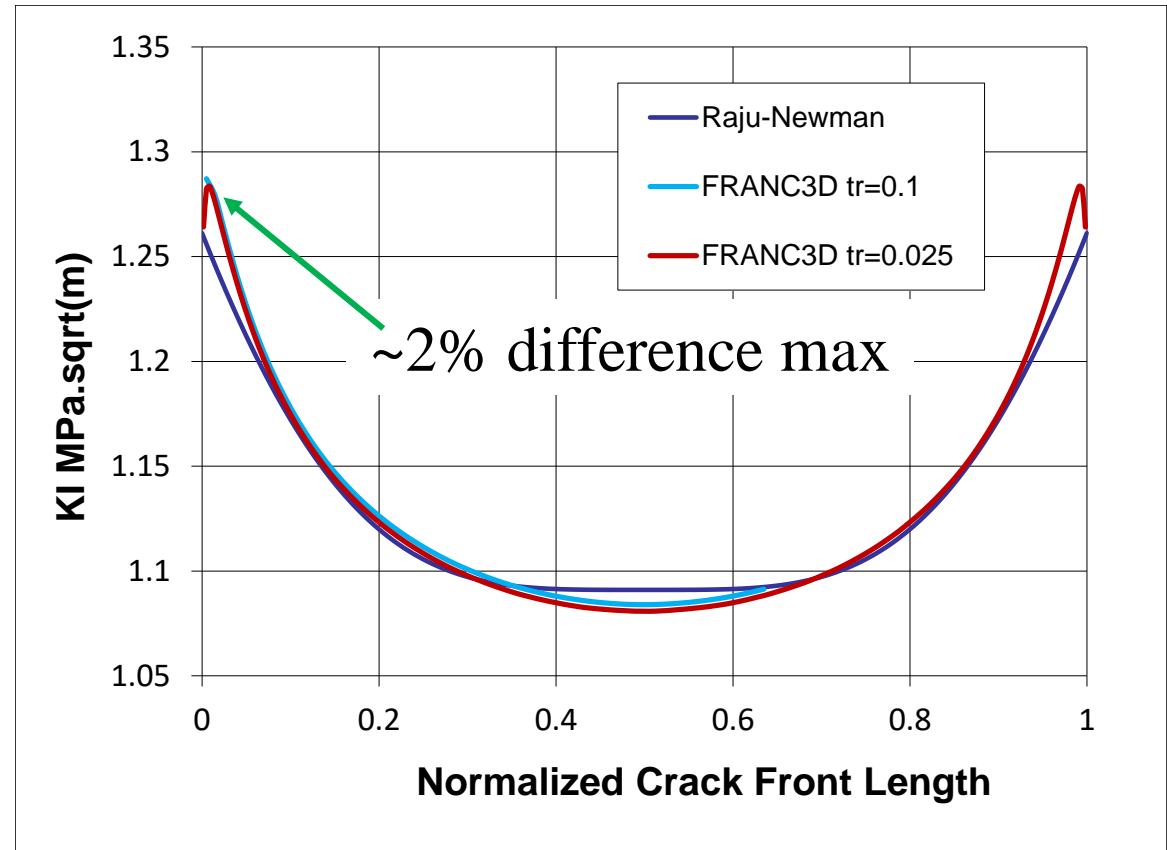
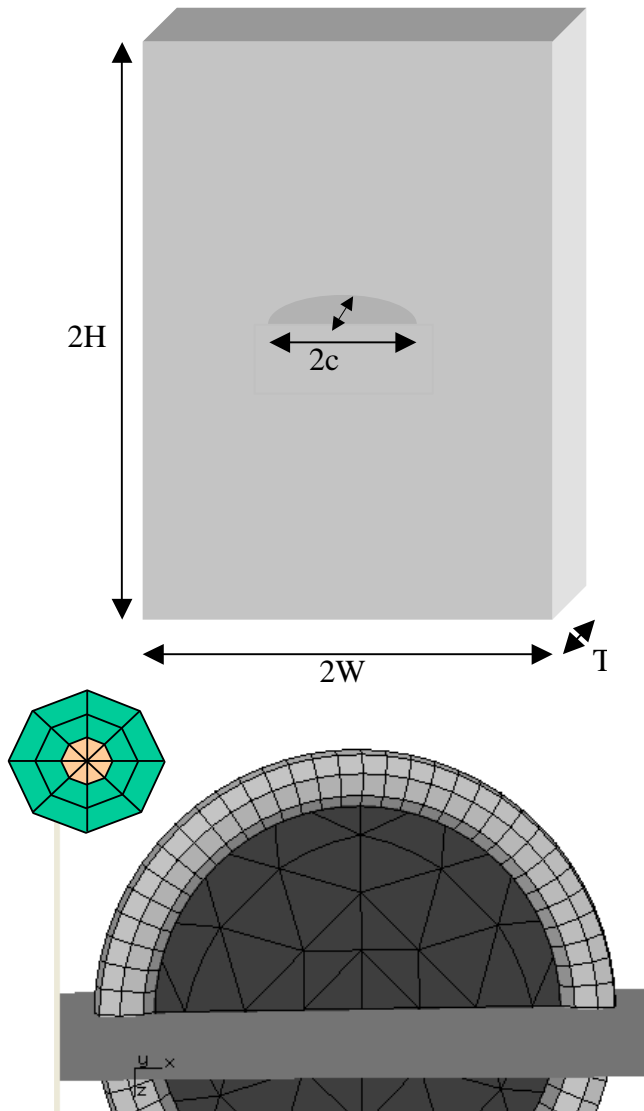


Template Cross-Section



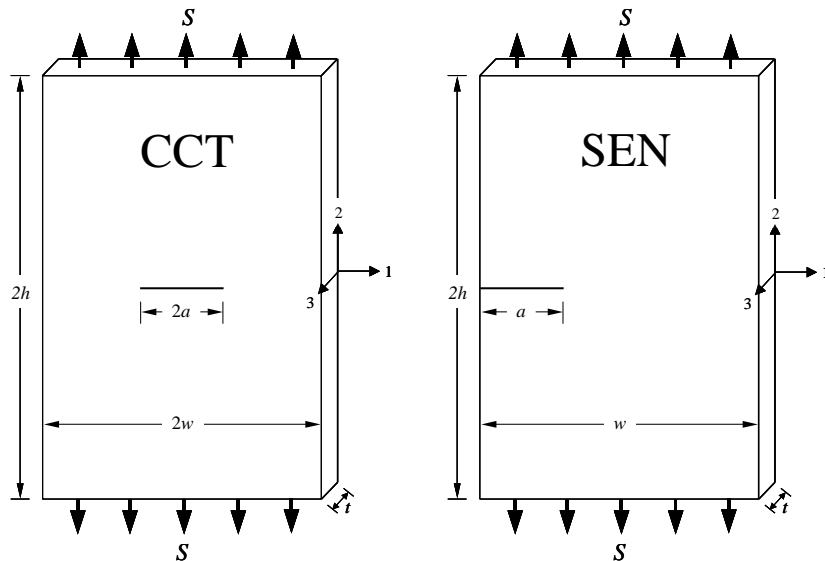
Default crack front mesh gives 0.2% error.

# M-Integral SIFs: Penny Surface Crack



The Raju-Newman solution is known to be slightly inaccurate near the free surface.

# Independent Mode I SIF Verification



Analyses performed by Dawn Phillips NASA Langley Research Center using FRANC3D Version 5.0

Users are strongly encouraged to do their own benchmark/validation.

**Table 2. Comparison of  $K_I$  (psi $\sqrt{in}$ ) for CCT specimen with various values of  $a/w$**

$a$	$a/w$	$K_{I, Ref.}$	$K_{I, Franc}$	% difference
0.5 in.	0.2	1.28	1.282	-0.16
1.0 in.	0.4	1.97	1.964	0.30
1.5 in.	0.6	2.82	2.825	-0.18
2.0 in.	0.8*	4.56	4.546	0.31

\* From Tada *et al.* solution only

**Table 3. Comparison of  $K_I$  (psi $\sqrt{in}$ ) for SEN specimen with various values of  $a/w$**

$a$	$a/w$	$K_{I, Ref.}$	$K_{I, Franc}$	% difference
1.0 in.	0.2	2.43	2.420	0.41
2.0 in.	0.4	5.29	5.284	0.11
3.0 in.	0.6	12.4	12.36	0.32
4.0 in.	0.8*	42.5	42.30	0.47

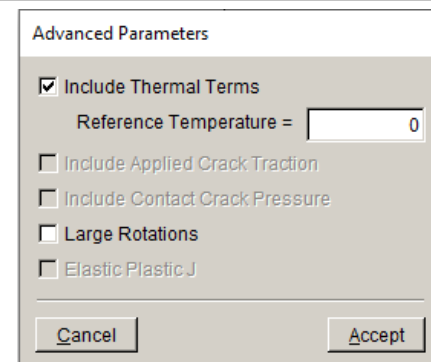
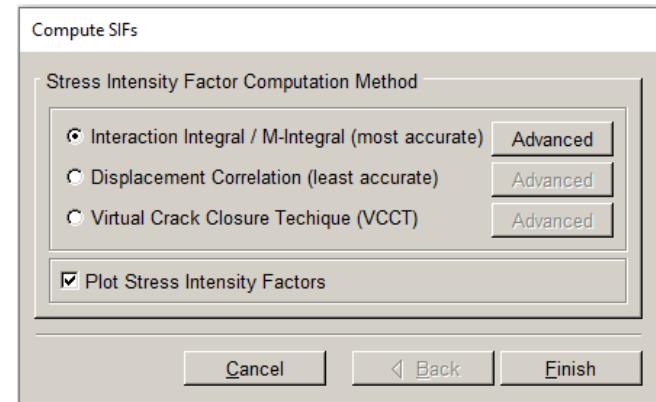
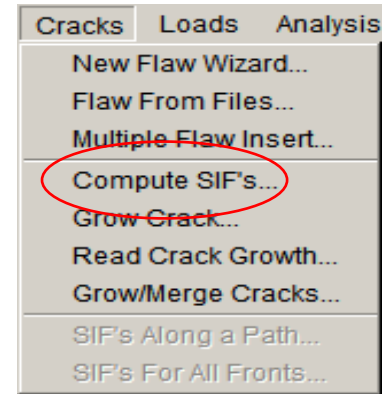
\* From Tada *et al.* solution only

# Computing SIFs

## FRANC3D User Interface

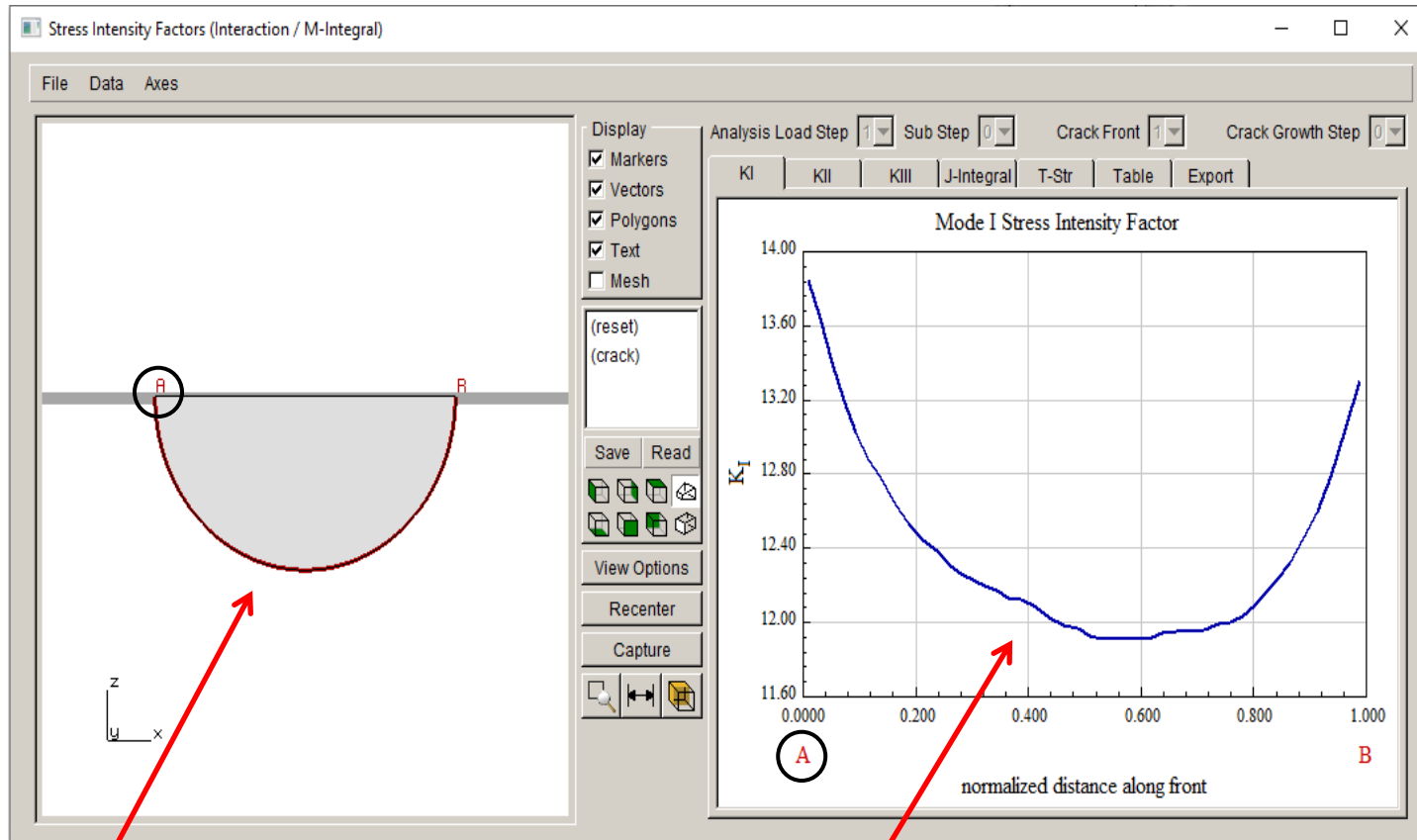
# Computing SIFs User Interface

- From the Crack Menu select Compute SIFs
- Compute SIFs dialog
  - Default in FRANC3D is M-integral
  - M-integral allows the user to include thermal terms, applied crack traction, contact crack pressure, large rotation (use **Advanced** button)
  - Displacement correlation can be used to verify the M-integral extra terms are included correctly
  - Virtual Crack Closure – added for cracks in bi-material interfaces
  - If you switch SIF computation method, that method will continue to be used until you switch back!
- Plot Stress Intensity Factors option turns on/off the display of the SIF plot dialog



# Plotting Options

- SIF distributions are displayed in this dialog

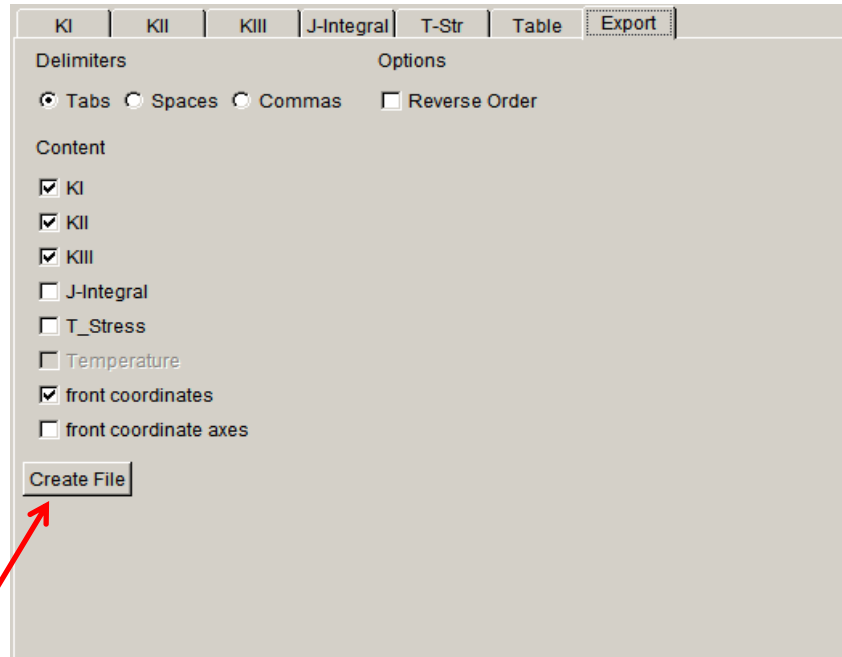


Crack in Model

Plot the computed SIF distributions along the crack front

# Plotting Options

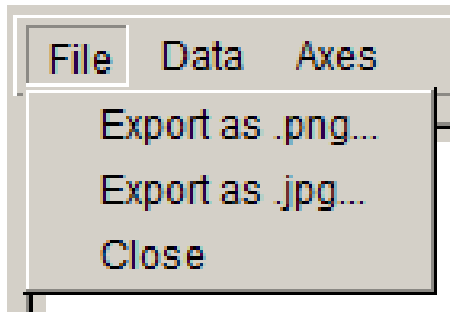
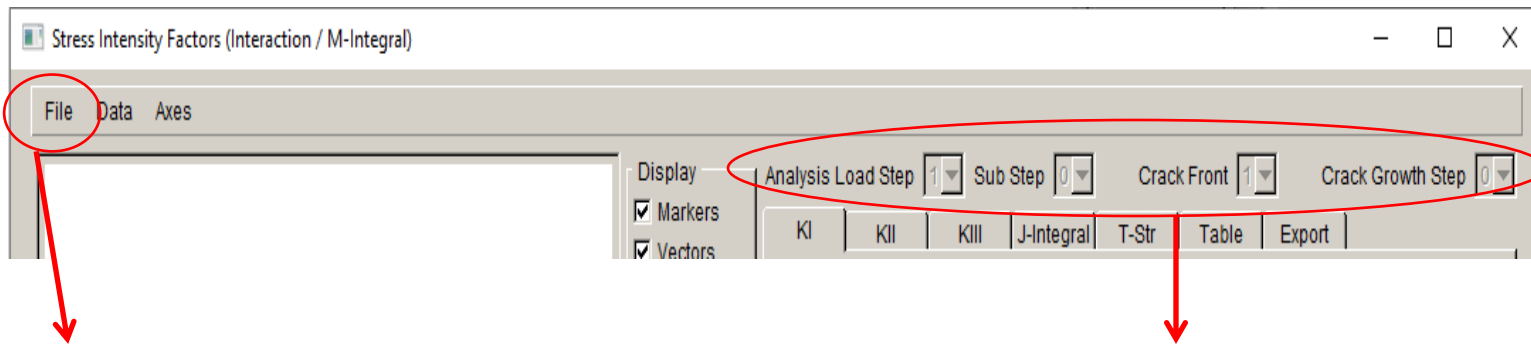
- Export tab allows the SIF data to be saved to a file



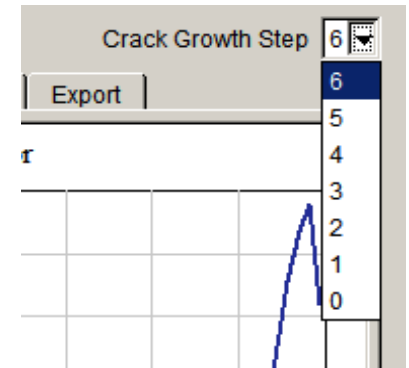
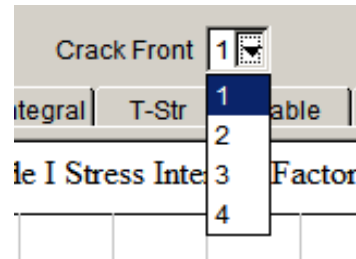
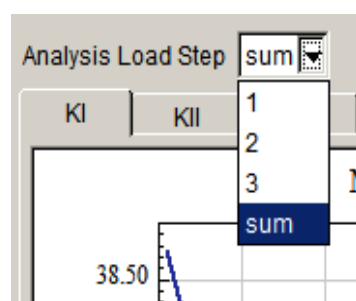
**Create File** button to select the file name and save the data

# Plotting Options

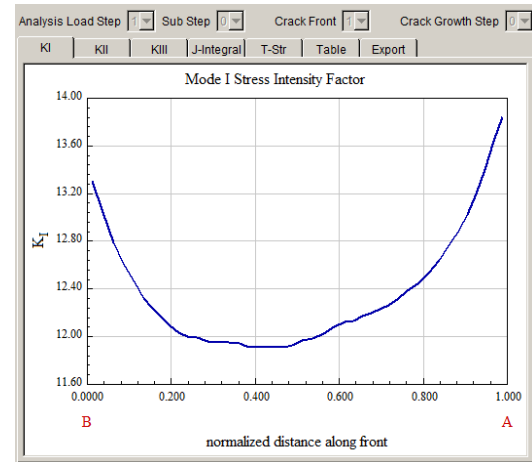
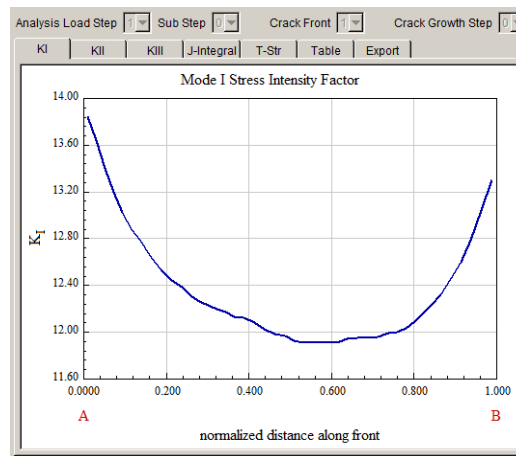
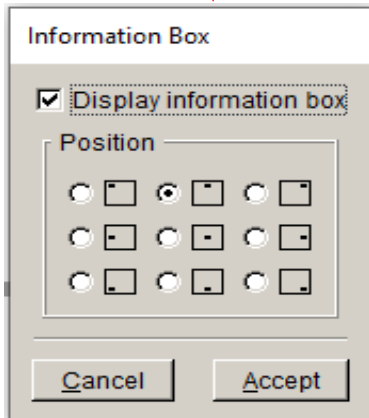
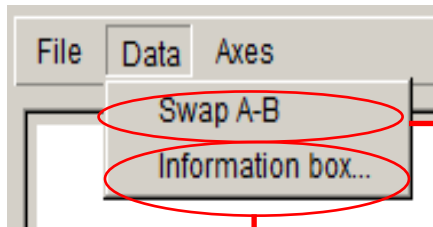
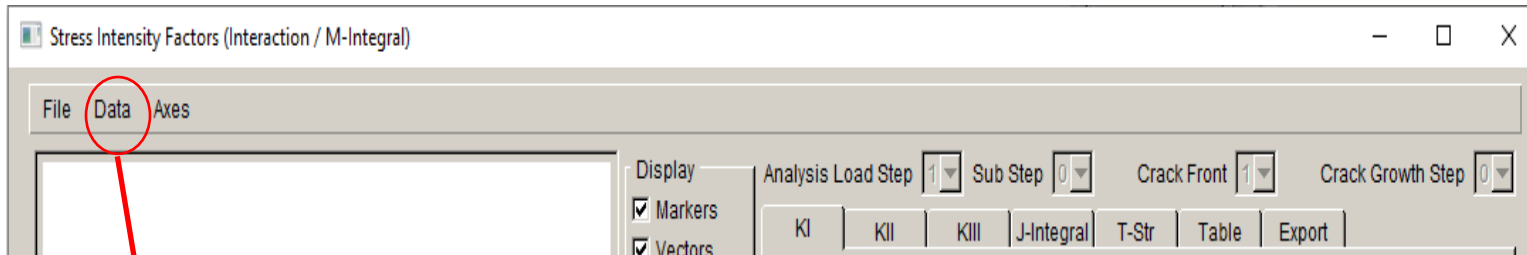
User can display SIF data from a model with multiple load steps (and substeps), multiple crack fronts, and multiple crack growth steps



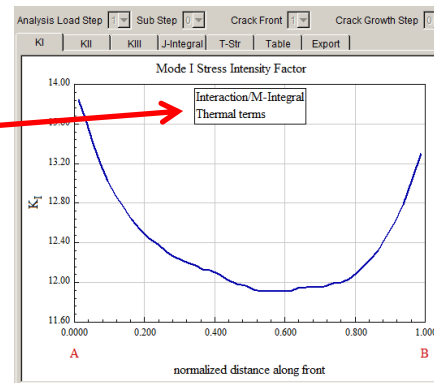
and export images of the XY plot to a file



# Plotting Options



Adds a text box on the XY plot



# Plotting Options

Stress Intensity Factors (Interaction / M-Integral)

File Data **Axes**

Display  
 Markers  
 Vectors

Analysis Load Step  Sub Step  Crack Front  Crack Growth Step

KI | KII | KIII | J-Integral | T-Str | Table | Export

File Data **Axes**

Format X Axis...  
Format Y Axis...  
Labels...

X Axis

Maximum:  Auto  Fixed   
Minimum:  Auto  Fixed   
MajorUnit:  Auto  Fixed   
Minor ticks per major tick:   
 Major tick lines  
 Minor tick lines

Cancel Accept

Labels

Title:   
X Axis:   
Y Axis:

Cancel Accept

Label syntax is described in Appendix A of the Reference

End Part 8