

FRANC3D Training Workshop: Part 8

SIFs from FE Analysis

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Workshop Agenda

- Part 1: Introduction to Fatigue and Damage Tolerance
- Part 2: Introduction to Fracture Mechanics Analysis
- Part 3: Introduction to FRANC3D
- Part 4: FRANC3D User Interface
- Part 5: Finite Element (FE) Model Import
- Part 6: Crack Insertion
- Part 7: Static Crack Analysis & SIF Computation
- **Part 8: SIFs from FE Analysis**
- Part 9: Crack Growth
- Part 10: SIF History & Fatigue Life
- Part 11: Miscellaneous Topics

SIFs from FE Analysis

- Linear Elastic Fracture Mechanics Using FE Analysis
- Stress Intensity Factors (SIFs)
- Computing Stress Intensity Factors
 - Displacement Correlation Method
 - Conservative Integral Method
 - J-Integral
 - M-Integral
- Benchmark Examples Using M-integral
- Computing SIFs FRANC3D User Interface

Linear Elastic Fracture Mechanics Analysis Using Finite Elements

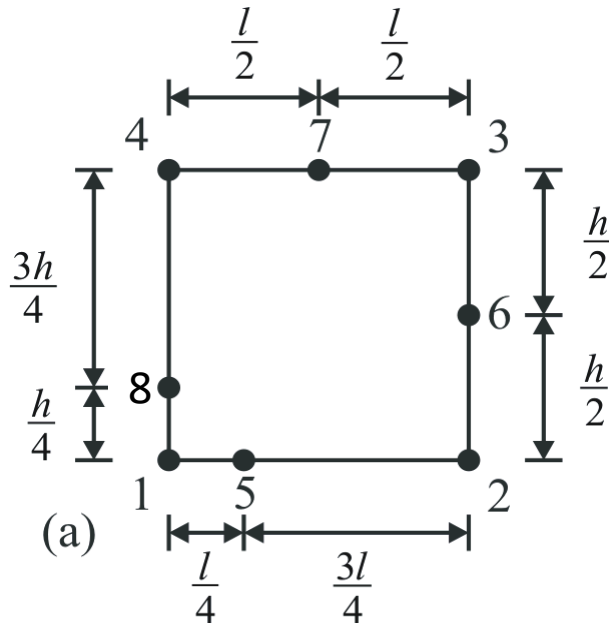
LEFM Using FEM

- Finite element method is a natural tool for performing LEFM problems
- Number of researchers investigated **special finite element formulations** that incorporate **singular basis functions or stress intensity factors as nodal variables**
 - These special elements are **not** available in most general-purpose finite element programs
- Finite Element Method (FEM) needs to be manipulated to handle the singular crack-tip stress and strain fields predicted by theory

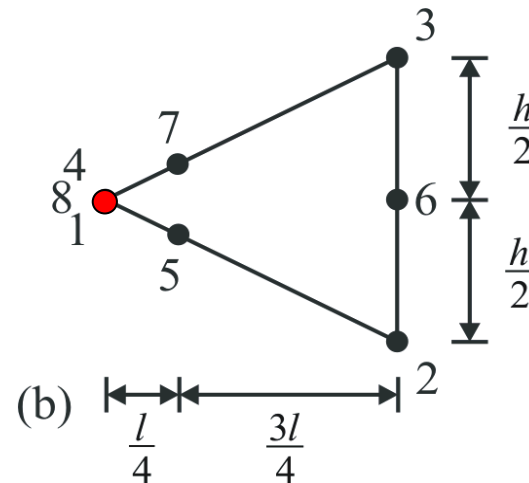
LEFM Using FEM

- A significant advancement in the use of the finite element method for LEFM problems was the simultaneous, and independent, development of the "quarter-point element".

Henshell and Shaw, 1975



Barsoum, 1976



Quadrilateral (a) and collapsed quadrilateral (b) quarter-point elements

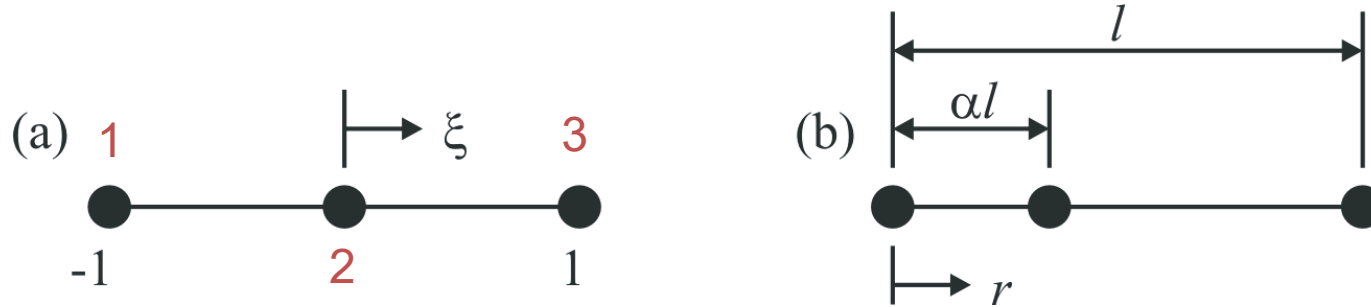
LEFM Using FEM

- These researchers showed that proper crack-tip displacement, stress, and strain fields are modeled by standard, quadratic order, isoparametric finite elements if one simply moves the element's mid-side node to the quarter-point position.
 - This introduces a singularity into the mapping between the element's parametric coordinate space and Cartesian space.

LEFM Using FEM

- The discovery of quarter-point elements was a significant milestone in the development of FE procedures for LEFM.
- With these elements, standard and widely available finite element programs can be used to model crack tip fields accurately with only **minimal preprocessing** required.

1D 1/4-Point Singular Element



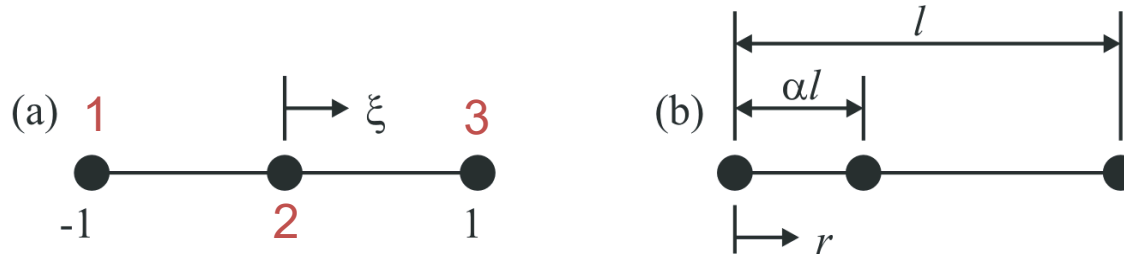
A quadratic element, where (a) shows the parametric space of the element, and (b) shows the Cartesian space of the element. The crack tip is at $r=0$.

$$\mathbf{u} = \sum_{i=1}^3 N_i \mathbf{u}_i = \frac{1}{2} \xi(\xi - 1) \mathbf{u}_1 + (1 - \xi^2) \mathbf{u}_2 + \frac{1}{2} \xi(\xi + 1) \mathbf{u}_3$$

The standard polynomial *displacement* interpolation scheme is regrouped:

$$\mathbf{u} = \mathbf{u}_2 + \frac{1}{2} (\mathbf{u}_3 - \mathbf{u}_1) \xi + \left(\frac{1}{2} (\mathbf{u}_1 + \mathbf{u}_3) - \mathbf{u}_2 \right) \xi^2$$

1D 1/4-Point Singular Element



$$r = \sum_{i=1}^3 N_i r_i = \alpha l + \frac{1}{2} l \xi + l \left(\frac{1}{2} - \alpha \right) \xi^2$$

and the standard, polynomial *geometry* interpolation scheme.

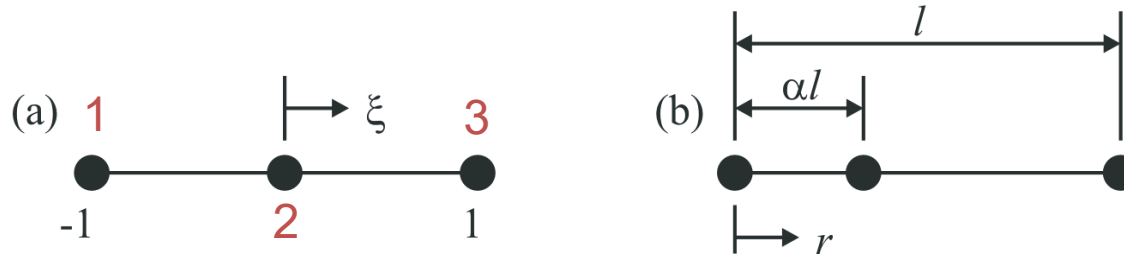
First, the usual case of mid-side geometry:

$$\alpha = \frac{1}{2} \longrightarrow \xi = \frac{2r}{l} - 1$$

The expected polynomial interpolation is:

$$u = u_1 + (-3u_1 + 4u_2 - u_3) \frac{r}{l} + 2(u_1 - 2u_2 + u_3) \frac{r^2}{l^2}$$

1D 1/4-Point Singular Element



$$r = \sum_{i=1}^3 N_i r_i = \alpha l + \frac{1}{2} l \xi + l \left(\frac{1}{2} - \alpha \right) \xi^2$$

The standard polynomial geometry interpolation scheme.

Next, the unusual case of 1/4-point geometry:

$$\alpha = \frac{1}{4} \longrightarrow \xi = \frac{2\sqrt{lr}}{l} - 1$$

The *unexpected* non-polynomial interpolation is:

$$u = u_1 + 2(u_1 - 2u_2 + u_3) \frac{r}{l} + (-3u_1 + 4u_2 + u_3) \frac{\sqrt{lr}}{l}$$

1D 1/4-Point Singular Element

Normal displacement field: $u = u_1 + (-3u_1 + 4u_2 - u_3) \frac{r}{l} + 2(u_1 - 2u_2 + u_3) \frac{r^2}{l^2}$

1/4-Point displacement field: $u = u_1 + 2(u_1 - 2u_2 + u_3) \frac{r}{l} + (-3u_1 + 4u_2 + u_3) \frac{\sqrt{lr}}{l}$

Compare to theoretical field: $u = \sum_{n=1}^{\infty} \frac{r^{\frac{n}{2}}}{2\mu} a_n^1 \left[\left(\kappa + \frac{n}{2} + (-1)^n \right) \cos \frac{n}{2} \theta - \frac{n}{2} \cos \left(\frac{n}{2} - 2 \right) \theta \right]$

Normal strain field: $\varepsilon = \frac{du}{dr} = (-3u_1 + 4u_2 - u_3) \frac{1}{l} + 4(u_1 - 2u_2 + u_3) \frac{r}{l^2}$

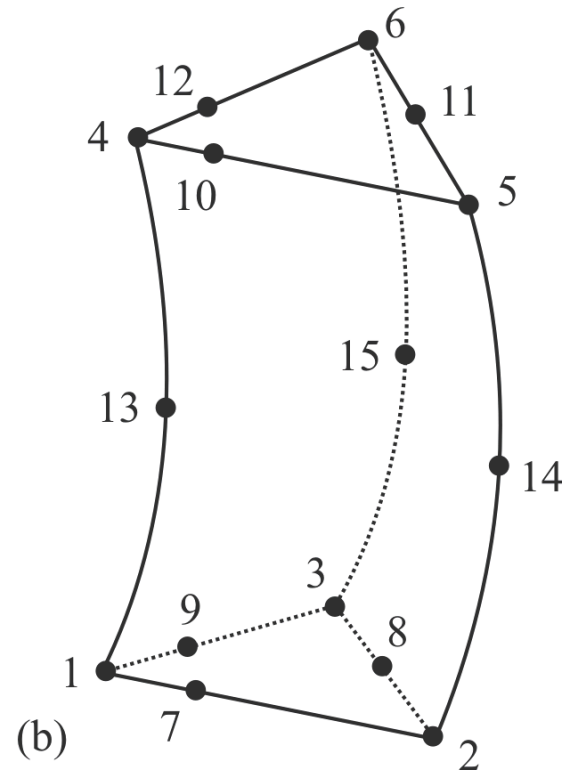
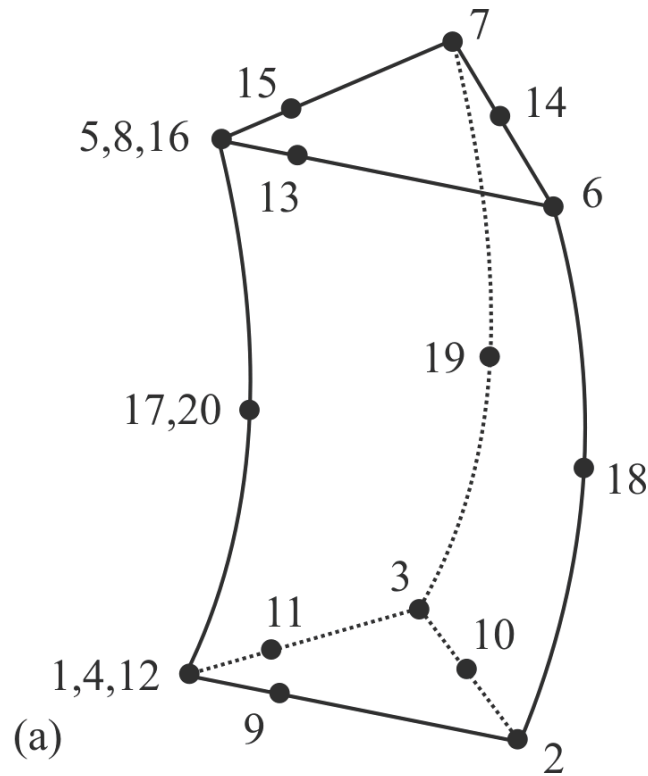
1/4-point strain field: $\varepsilon = \frac{du}{dr} = 2(u_1 - 2u_2 + u_3) \frac{1}{l} + \left(-\frac{3}{2}u_1 + 2u_2 - \frac{1}{2}u_3 \right) \frac{1}{\sqrt{lr}}$

Compare to theoretical field: $\sigma_x = \sum_{n=1}^{\infty} \frac{n}{2} r^{\frac{n}{2}-1} a_n^1 \left[\left(2 + \frac{n}{2} + (-1)^n \right) \cos \left(\frac{n}{2} - 1 \right) \theta - \left(\frac{n}{2} - 1 \right) \cos \left(\frac{n}{2} - 3 \right) \theta \right]$

1D $\frac{1}{4}$ -Point Singular Element

- Displacement field:
 - Contains a constant value, a linear variation in r , and the square root variation in r .
 - Corresponds to the leading terms in the LEFM expressions for the near crack-tip displacement.
- Strain field:
 - Contains a constant term and a singular term that varies as $r^{-1/2}$
 - Corresponds to the leading term in the LEFM stress and strain expansions

3D $\frac{1}{4}$ -Point Singular Elements



- (a) collapsed, 20-noded brick, quarter-point element
(b) natural 15-noded, wedge quarter-point element

Stress Intensity Factors

Stress Intensity Factors

- SIF's are a measure of the intensity of the stress concentration along a crack front
- Range in the SIF (ΔK) due to cyclic loading can be correlated with the crack growth rate
- SIF's are functions of the crack size and shape, local geometry of the component, and local stresses in the crack region

Stress Intensity Factors

- FRANC3D was developed primarily to calculate highly **accurate** SIF's for **arbitrarily** shaped cracks in **arbitrarily** shaped components subjected **arbitrary** loading
- FRANC3D **handles real-world conditions** such as non-planar cracks, cracks with complex shapes, complex component geometries, contact between components, residual stresses, and local stress concentrations

Stress Intensity Factors

- FRANC3D computes stress intensity factors for all three “modes” of fracture at FE mid-side nodes along the crack front
- Under conditions of small-scale yielding, crack behavior is controlled by stress intensity factors:
 - Stability – will the crack tip move?
 - Trajectory – in what direction?
 - Rate – how fast?

Computing Stress Intensity Factors

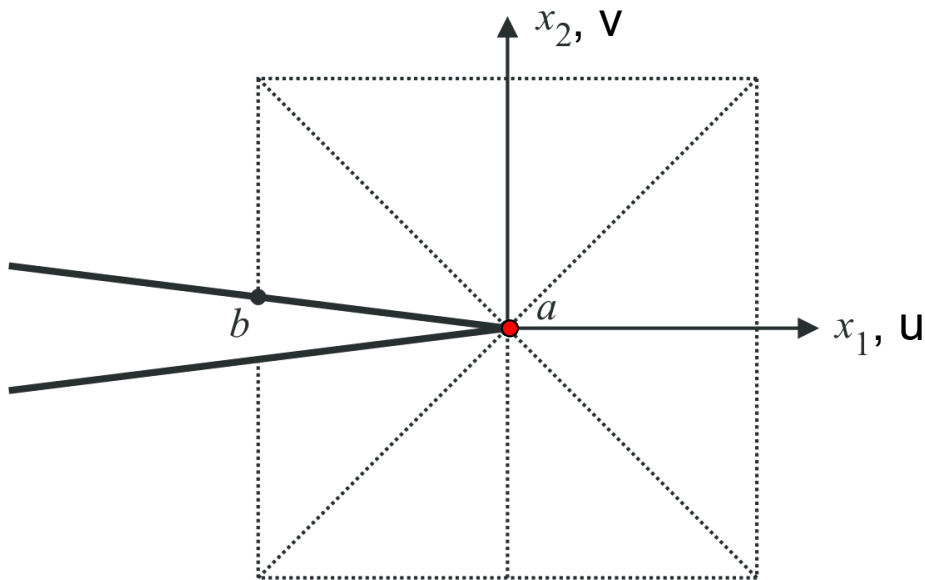
- FRANC3D has three methods of computing stress intensity factors (SIF's or K's):
 - Displacement Correlation Method
 - Conservative Integral Method
 - J -Integral
 - M -Integral
 - Virtual Crack Closure Method

Displacement Correlation Method

Displacement Correlation Method

- Relatively simple to understand and implement
- Relatively poor accuracy ($\sim 5\%$ error for a reasonable mesh)
- Good sanity check but not for production work

Theoretical asymptotic displacement fields



Set $r = r_{a-b}$, and $\theta = 180^\circ$

$$v_b - v_a = \frac{K_I}{\mu} \left[\frac{r_{a-b}}{2\pi} \right]^{1/2} (2 - 2\nu)$$

$$u = \frac{K_I}{\mu} \left[\frac{r}{2\pi} \right]^{1/2} \cos \frac{\theta}{2} \left[1 - 2\nu + \sin^2 \frac{\theta}{2} \right]$$

$$v = \frac{K_I}{\mu} \left[\frac{r}{2\pi} \right]^{1/2} \sin \frac{\theta}{2} \left[2 - 2\nu - \cos^2 \frac{\theta}{2} \right]$$

Note: for plane stress, let $\nu = \nu/(1 + \nu)$

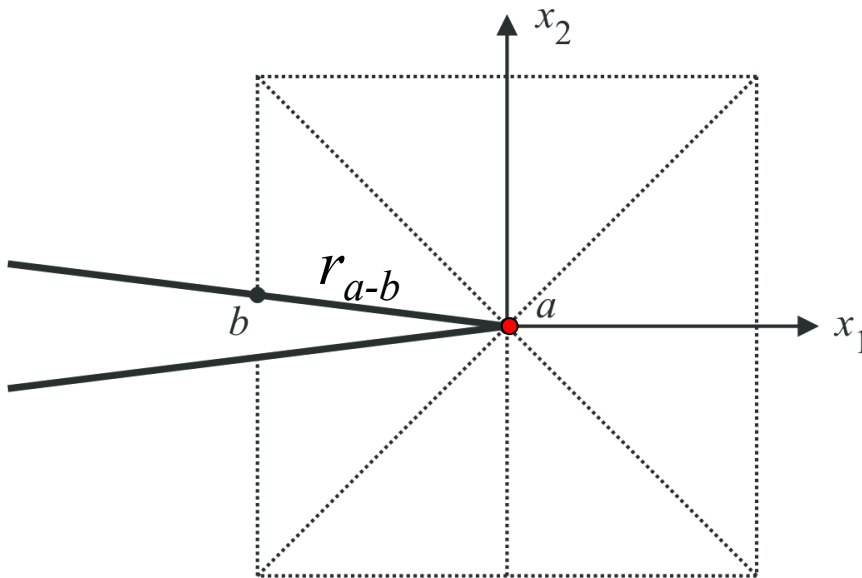
$$u = \frac{K_{II}}{\mu} \left[\frac{r}{2\pi} \right]^{1/2} \sin \frac{\theta}{2} \left[2 - 2\nu + \cos^2 \frac{\theta}{2} \right]$$

$$v = \frac{K_{II}}{\mu} \left[\frac{r}{2\pi} \right]^{1/2} \cos \frac{\theta}{2} \left[-1 + 2\nu + \sin^2 \frac{\theta}{2} \right]$$

$$u_b - u_a = \frac{K_{II}}{\mu} \left[\frac{r_{a-b}}{2\pi} \right]^{1/2} (2 - 2\nu)$$

Displacement Correlation (No Crack-Front Template)

For plane strain case:



$$K_I = \frac{\mu\sqrt{2\pi}(v_b - v_a)}{\sqrt{r_{a-b}}(2 - 2\nu)}$$

$$K_{II} = \frac{\mu\sqrt{2\pi}(u_b - u_a)}{\sqrt{r_{a-b}}(2 - 2\nu)}$$

$$K_{III} = \frac{\mu\sqrt{\pi}(w_b - w_a)}{\sqrt{2r_{a-b}}}$$

μ is the shear modulus, ν is Poisson's ratio, r is the distance from the crack tip to the correlation point, and u_i , v_i , w_i are the x , y , and z displacements at point i

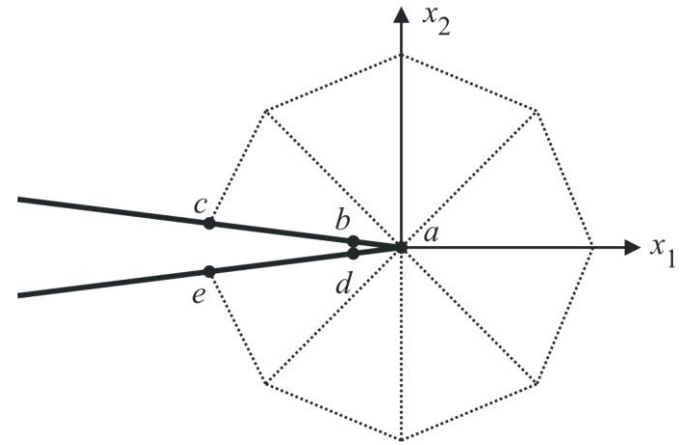
Same expressions used for plane stress assumptions if ν is replaced by $\nu / (1 + \nu)$

Displacement Correlation (With Crack-Front Template)

- SIF's computed by this approach can be improved if quarter-point crack-tip elements are used
- Displacement field:

$$v_{upper} = v_a + (-3v_a + 4v_b - v_c)\sqrt{\frac{r}{l}} + (2v_a - 4v_b + 2v_c)\frac{r}{l}$$

$$v_{lower} = v_a + (-3v_a + 4v_d - v_e)\sqrt{\frac{r}{l}} + (2v_a - 4v_d + 2v_e)\frac{r}{l}$$



The FE crack opening displacement (COD) is

$$v_{upper} - v_{lower} = [4(v_b - v_d) + v_e - v_c]\sqrt{\frac{r}{l}} + [4(v_b - v_d) + 2(v_c - v_e)]\frac{r}{l}$$

Displacement Correlation (With Crack-Front Template)

- The square root term of the FE COD can then be substituted into the analytical crack-tip displacement field to yield

$$K_I = \frac{\mu\sqrt{2\pi}}{\sqrt{r}(2-2\nu)} [4(v_b - v_d) + v_e - v_c]$$

$$K_{II} = \frac{\mu\sqrt{2\pi}}{\sqrt{r}(2-2\nu)} [4(u_b - u_d) + u_e - u_c]$$

Conservative Integral Methods

M- & *J*-Integrals

J -Integral

- In linear Elastic Fracture Mechanics (LEFM), the J -integral is equivalent to the fracture energy release rate, G .

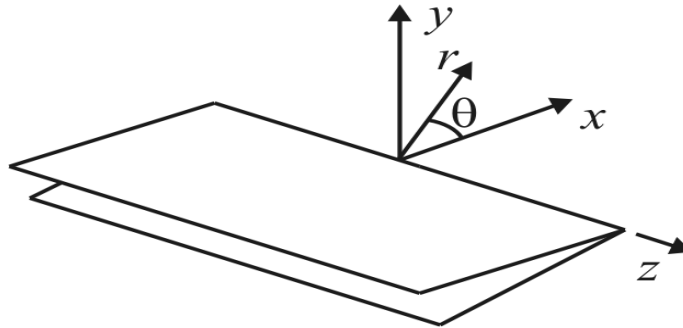
Energy Release Rate, G

- G can be computed either:
 - Using the stress intensity factors and material properties and the theoretical expressions for near crack-front displacements and stresses
- OR*
- Using the near-crack-front stresses, strains and displacement determined from a finite element analysis (domain integration)

G Computation Using the Stress Intensity Factors and Material Properties

Energy Release Rate Expression Using SIFs & Material Properties

Given the following crack-front coordinate systems (Rectangular and Polar coordinate systems)



The crack-tip energy release rates can be determined from Irwin's* crack closure integral

$$G = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \int_0^{\Delta} \sigma_{yi}(\Delta - r, 0) \cdot u_i(r, \pi) dr, \quad i = x, y, z \quad (1)$$

where $\sigma(r, \theta)$ are the near crack-front stresses (in cylindrical coordinates), and $u(r, \theta)$ are the near crack-front displacements

* Irwin, G.R. (1957) Analysis of stresses and strains near the end of a crack traversing a plate, *Journal of Applied Mechanics*, 24, 361-364

Energy Release Rate Expression Using SIFs & Material Properties

The first term of Williams* expansion for the stresses and displacements near the crack tip is related to the stress intensity factors and gives the following expression for stress and displacement assuming plane strain conditions.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{1}{\sqrt{2\pi r}} \begin{bmatrix} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) & \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}\right) & 0 \\ \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) & \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} & 0 \\ \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} & \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}\right) & 0 \\ 0 & 0 & -\sin \frac{\theta}{2} \\ 0 & 0 & \cos \frac{\theta}{2} \end{bmatrix} \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix} \quad (2)$$

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \frac{2(1+\nu)}{E} \sqrt{\frac{r}{2\pi}} \begin{bmatrix} \cos \frac{\theta}{2} \left(1 - 2\nu + \sin^2 \frac{\theta}{2}\right) & \sin \frac{\theta}{2} \left(2 - 2\nu + \cos^2 \frac{\theta}{2}\right) & 0 \\ \sin \frac{\theta}{2} \left(2 - 2\nu - \cos^2 \frac{\theta}{2}\right) & \cos \frac{\theta}{2} \left(2\nu - 1 + \sin^2 \frac{\theta}{2}\right) & 0 \\ 0 & 0 & 2\sin \frac{\theta}{2} \end{bmatrix} \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix} \quad (3)$$

* Williams, M.L. (1957) On the stress distribution at the base of a stationary crack, *Journal of Applied Mechanics*, 24, 109-114

Energy Release Rate Expression Using SIFs & Material Properties

Substituting $\theta = 0$ into equation 2, and $\theta = \pi$ in equation 3 gives

$$\begin{Bmatrix} \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \end{Bmatrix} = \frac{1}{\sqrt{2\pi r}} \begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \frac{2(1+\nu)}{E} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} 2(1-\nu)K_I \\ 2(1-\nu)K_{II} \\ K_{III} \end{Bmatrix} \quad (4)$$

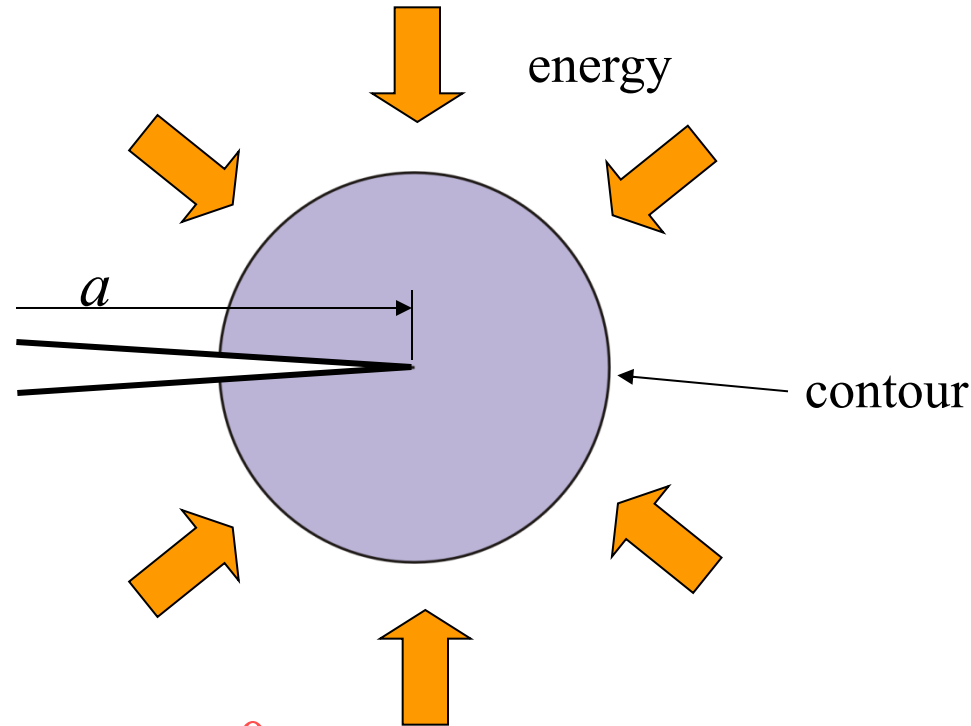
Substituting equations 4 into 1, and evaluating the integral gives the well know result

$$G = \frac{1-\nu^2}{E} K_I^2 + \frac{1-\nu^2}{E} K_{II}^2 + \frac{1+\nu}{E} K_{III}^2 \quad (5)$$

the energy release rate in terms of stress intensity factors and material properties

G Computation Using the near-crack-front stresses, strains and displacement determined from a finite element analysis (domain integration)

2D Crack-Tip Energy Balance (J-integral)



informally:

$$\frac{dE_{contour}}{da} = \frac{dE_{elastic}}{da} + \frac{dE_{plastic}}{da} + \frac{dE_{newsurface}}{da} + \frac{dE_{heat}}{da} + \frac{dE_{sound}}{da} + \frac{dE_{light}}{da}$$

0
small

which reduces to:

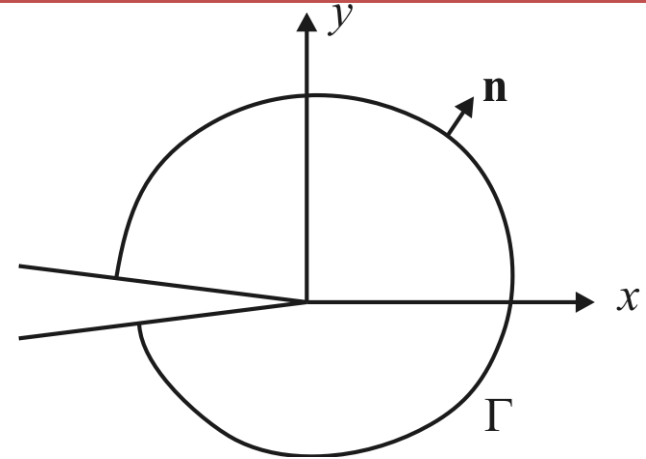
$$-\frac{dE_{newsurface}}{da} = G = \frac{dE_{elastic}}{da} - \frac{dE_{contour}}{da} = J$$

Energy Release Rate Expressed Using Near-Crack-front Stresses, Stains and Displacements

The J -Integral* measures energy flux into the crack-tip region

$$J = \int_{\Gamma} \left(W n_x - T_i \frac{\partial u_i}{\partial x} \right) ds \quad W = \frac{1}{2} \sigma_{ij} \epsilon_{ij}$$

Under conditions of small scale yielding, **J -Integral is equal to energy release rate, G**



The contour J -Integral can be recast as an equivalent area (volume in 3D) integral**, which is more **accurate and stable** in a finite element context

$$J' = \int_V \left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right) \frac{\partial q}{\partial x_j} dv \quad (6)$$

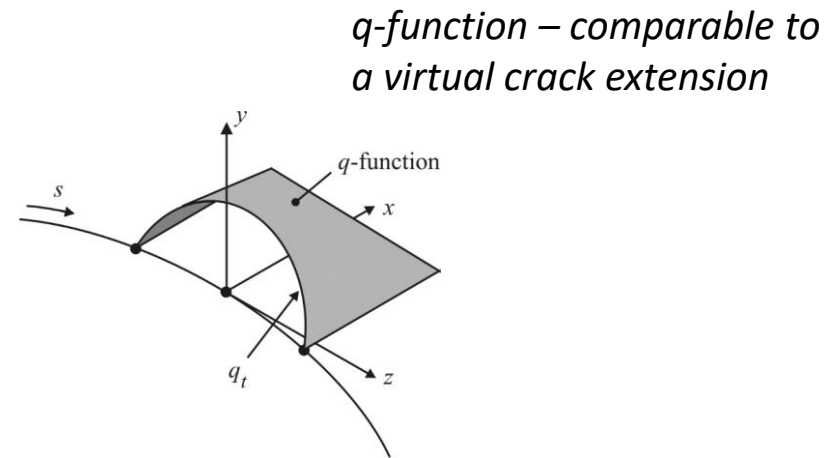
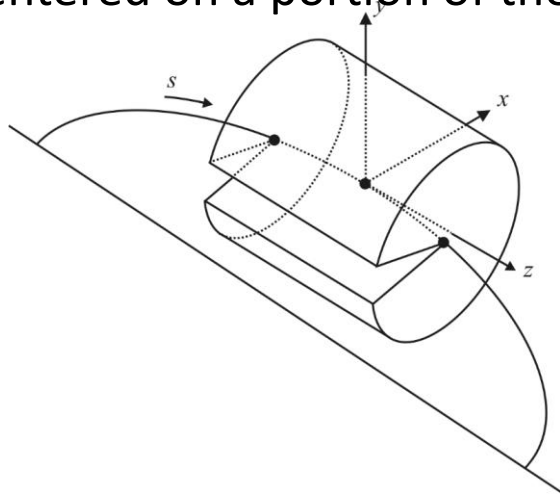
q is a function that is one at the crack tip and zero on the boundary of the integration domain. It can be interpreted as a virtual crack extension, and δ is the Kronecker delta

* Rice, J.R. (1968) A path independent integral and approximate analysis of strain concentrations by notches and cracks, *Journal of Applied Mechanics*, 35, 379-386

** Li, F.Z., Shih, C.F., and Needleman, A. (1985) A comparison of methods for calculating energy release rates, *Engineering Fracture Mechanics*, 21, 405-421

Energy Release Rate Expressed Using Near-Crack-front Stresses, Strains and Displacements

In 3D, the J-Integral is evaluated within a cylindrical domain centered on a portion of the crack-front.



J' is the energy release for the specified virtual crack extension q . This must be normalized to get the energy released for a unit amount of new crack area

$$G = \frac{J'}{\int q(s) ds} = \frac{J'}{A_q} = \int_V \left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right) \frac{\partial q}{\partial x_j} dV \bigg/ A_q \quad (7)$$

This is an expression for the energy release rate in terms of stresses, strains, and displacements

J -Integral

- The problem with J is that it only gives **one number** for the fracture energy release rate
 - Difficult to segregate the fracture energy release rate among the three modes of fracture.

M -Integral

(also called the interaction integral)

Computing Stress Intensity Factors

M-Integral (Interaction Integral):

- Numerically, M -Integral is similar to J -Integral
- M -Integral computes strain energy release rates (G_I, G_{II}, G_{III}) and stress intensity factors (K_I, K_{II}, K_{III}) associated with three modes of fracture
 - Mode II (K_{II}) needed to predict kink angle for crack growth direction
- M -integral implementation in FRANC3D works for isotropic and anisotropic materials
- Relatively good accuracy (<1% error for a reasonable mesh density...no need for super-refined mesh at the crack front)
- Requires additional terms for crack face tractions, residual stresses, thermal strains, etc.

¹ Wawrzynek, P.A., Carter, B., and Banks-Sills, L. "The M -integral for computing stress intensity factors in generally anisotropic materials," NASA/CR-2005-214006. July 2005

² Banks-Sills, L., Wawrzynek, P.A., Carter, B., Ingraffea, T.R., and Hershkowitz, I., "Methods for computing stress intensity factors in anisotropic geometries: Part II – arbitrary geometry," *Engng. Fracture Mech*, Vol 74, No 8, pp 1293-1307. July 2006

Steps for Formulating the M-Integral

- **Step(1):** Invoke superposition to combine the finite element solution with a known analytical solution
- **Step(2):** Substitute the superposition terms into the energy release rate (G) equations (5) and (7) from the previous slides
- **Step(3):** Equate the two expressions for G and substitute a simple analytical K_I solution, and solve for the finite element K_I
- **Step (4):** Repeat step (3) for K_{II} and K_{III}

Principal of Superposition

The principal of superposition for elasticity says that if you have one solution that satisfies the governing equations for elasticity, and you add it to a second solution that satisfies these equations, the result will also satisfy the governing equations

Therefore,

$$K_I = K_I^{(1)} + K_I^{(2)}, \quad K_{II} = K_{II}^{(1)} + K_{II}^{(2)}, \quad K_{III} = K_{III}^{(1)} + K_{III}^{(2)} \quad (8)$$

$$\sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}, \quad \varepsilon_{ij} = \varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}, \quad u_i = u_i^{(1)} + u_i^{(2)} \quad (9)$$

where the $(^1)$ solution is the finite element results
and $(^2)$ is an analytical solution

Substitution in The Energy Release Rate Equations

Substituting equation (8) into equation (5) (*energy release rate in terms of stress intensity factors and material properties*), and collecting terms gives

$$G = G^{(1)} + G^{(2)} + M^{(1,2)} \quad (10)$$

where

$$M^{(1,2)} = \frac{1-\nu^2}{E} K_I^{(1)} K_I^{(2)} + \frac{1-\nu^2}{E} K_{II}^{(1)} K_{II}^{(2)} + \frac{1+\nu}{E} K_{III}^{(1)} K_{III}^{(2)} \quad (11)$$

Substitution in The Energy Release Rate Equations

Substituting equation (9) into equation (7) (*energy release rate in terms of the near-crack-front stresses, strains and displacements*), and collecting terms gives

$$G = G^{(1)} + G^{(2)} + M^{(1,2)} \quad (12)$$

Where in this case

$$M^{(1,2)} = \int_V \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} dV \bigg/ A_q \quad (13)$$

Solving for The Finite Element $K_I^{(1)}$

For the (2) solution, we use the first term of the Williams expansion, (equations 2 and 3) after setting $K_I = 1$ and $K_{II} = K_{III} = 0$. Substituting this into equation 11 and 13, and setting the two expressions for M equal, gives

$$\frac{1-\nu^2}{E} K_I^{(1)} = \int_V \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} dV \Bigg/ A_q \quad (14)$$

or, the finite element K_I is

$$K_I^{(1)} = \frac{E}{1-\nu^2} \int_V \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} dV \Bigg/ A_q \quad (15)$$

where $\sigma^{(1)}$ and $u^{(1)}$ come from finite element results, and $\sigma^{(2)}$ and $u^{(2)}$ are (next slide)

Solving for The Finite Element $K_I^{(1)}$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{1}{\sqrt{2\pi r}} \begin{Bmatrix} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\ \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\ \nu(\sigma_{xx} + \sigma_{yy}) \\ \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \\ 0 \\ 0 \end{Bmatrix} \quad (16)$$

and

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \frac{2(1+\nu)}{E} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos \frac{\theta}{2} \left(1 - 2\nu + \sin^2 \frac{\theta}{2} \right) \\ \sin \frac{\theta}{2} \left(2 - 2\nu - \cos^2 \frac{\theta}{2} \right) \\ 0 \end{Bmatrix} \quad (17)$$

These come from equations 2 and 3 with $K_I = 1$ and $K_{II} = K_{III} = 0$

Solving for The Finite Element $K_{II}^{(1)}$

For the $(^2)$ solution, we use the first term of the Williams expansion, (equations 2 and 3) after setting $K_I = 0$, $K_{II} = 1$ and $K_{III} = 0$. Substituting this into equation 11 and 13, and setting the two expressions for M equal, the finite element K_{II} is

$$K_{II}^{(1)} = \frac{E}{1-\nu^2} \int_V \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} dV \bigg/ A_q \quad (18)$$

where $\sigma^{(1)}$ and $u^{(1)}$ come from finite element results, and $\sigma^{(2)}$ and $u^{(2)}$ are (next slide)

Solving for The Finite Element $K_{II}^{(1)}$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{1}{\sqrt{2\pi r}} \begin{Bmatrix} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \\ \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \\ \nu(\sigma_{xx} + \sigma_{yy}) \\ \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\ 0 \\ 0 \end{Bmatrix} \quad (19)$$

and

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \frac{2(1+\nu)}{E} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \sin \frac{\theta}{2} \left(2 - 2\nu + \cos^2 \frac{\theta}{2} \right) \\ \cos \frac{\theta}{2} \left(-1 + 2\nu + \sin^2 \frac{\theta}{2} \right) \\ 0 \end{Bmatrix} \quad (20)$$

These come from equations 2 and 3 with $K_I = 0$, $K_{II} = 1$ and $K_{III} = 0$

Solving for The Finite Element $K_{III}^{(1)}$

For the $(^2)$ solution, we use the first term of the Williams expansion, (equations 2 and 3) after setting $K_I = K_{II} = 0$ and $K_{III} = 1$. Substituting this into equation 11 and 13, and setting the two expressions for M equal, the finite element K_{III} is

$$K_{III}^{(1)} = \frac{E}{1+\nu} \int_V \left(\sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} + \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right) \frac{\partial q}{\partial x_j} dV \bigg/ A_q \quad (21)$$

where $\sigma^{(1)}$ and $u^{(1)}$ come from finite element results, and $\sigma^{(2)}$ and $u^{(2)}$ are (next slide)

Solving for The Finite Element $K_{III}^{(1)}$

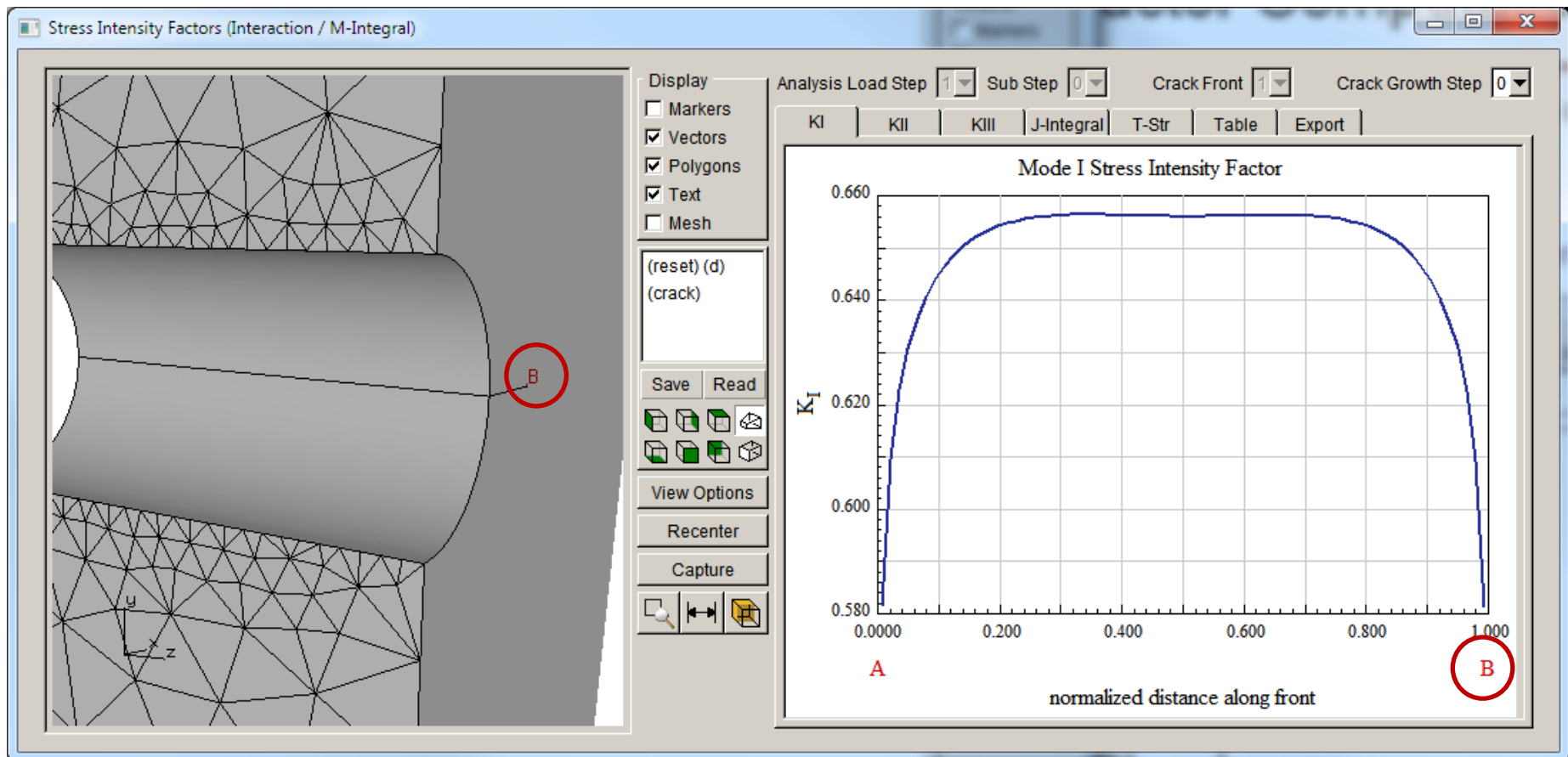
$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{1}{\sqrt{2\pi r}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \frac{2(1+\nu)}{E} \sqrt{\frac{r}{2\pi}} \begin{Bmatrix} 0 \\ 0 \\ \sin \frac{\theta}{2} \end{Bmatrix} \quad (22)$$

These come from equations 2 and 3 with $K_I = 0$, $K_{II} = 0$ and $K_{III} = 1$

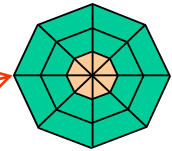
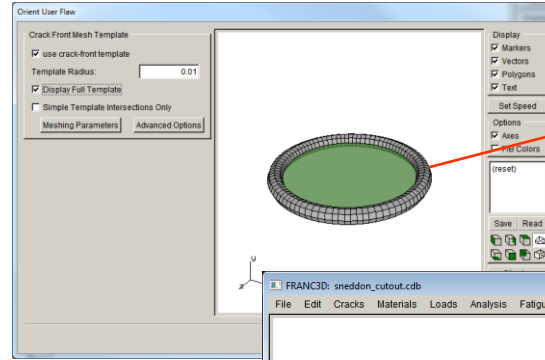
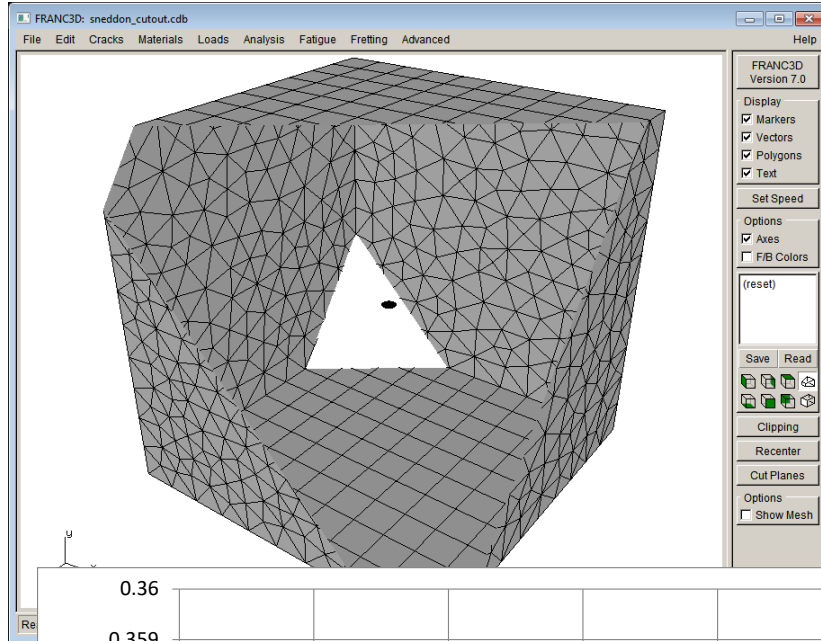
Benchmark Examples Using M-integral

Stress Intensity Factor Computations

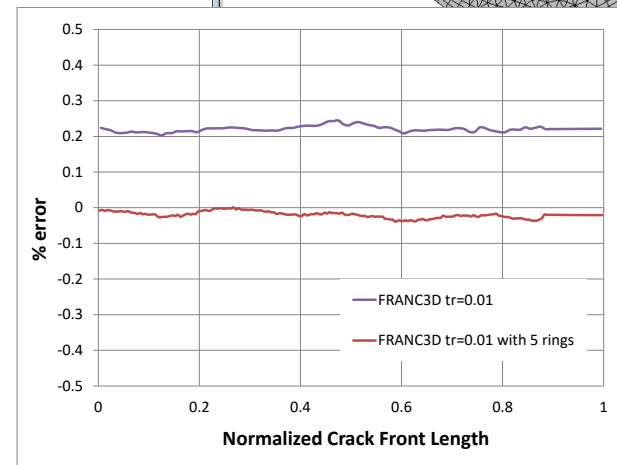
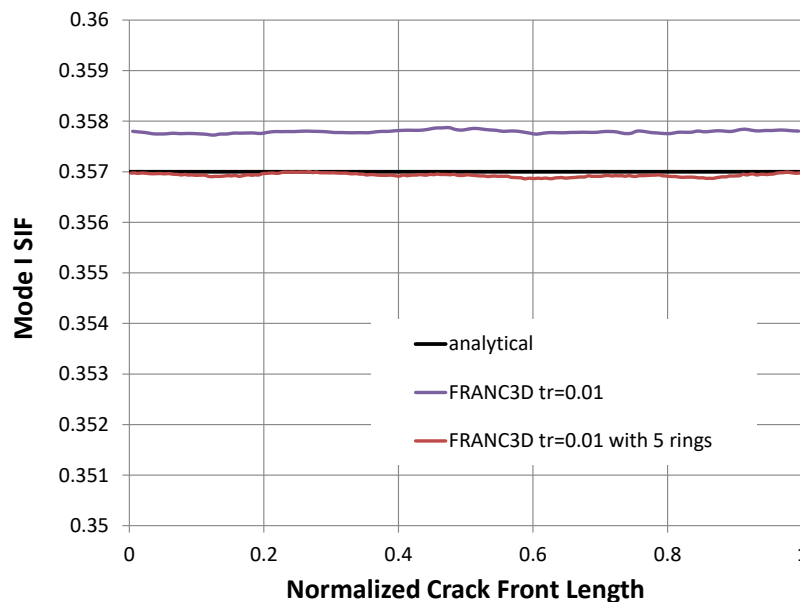
FRANC3D displays stress intensity factors along the normalized (from 0 to 1) crack front length, starting at **A** and ending at **B**



M-Integral SIFs: Embedded Penny Crack

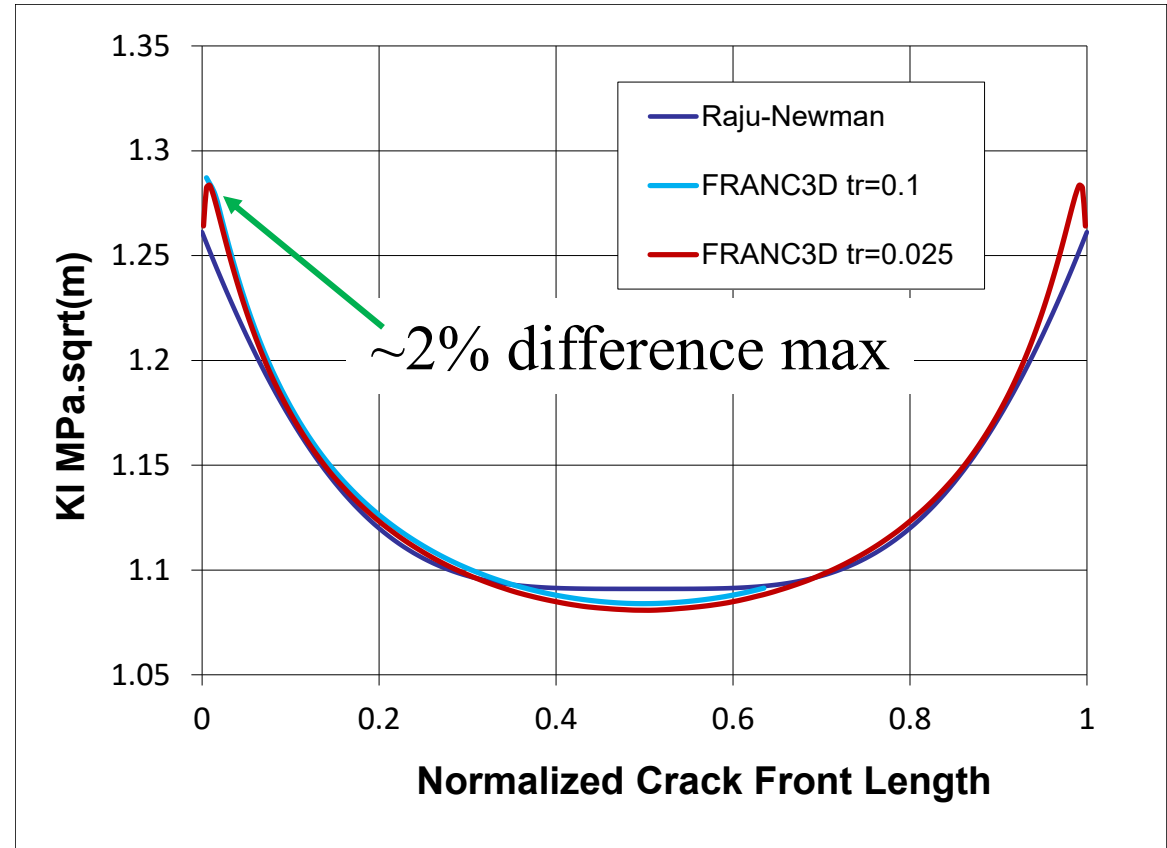
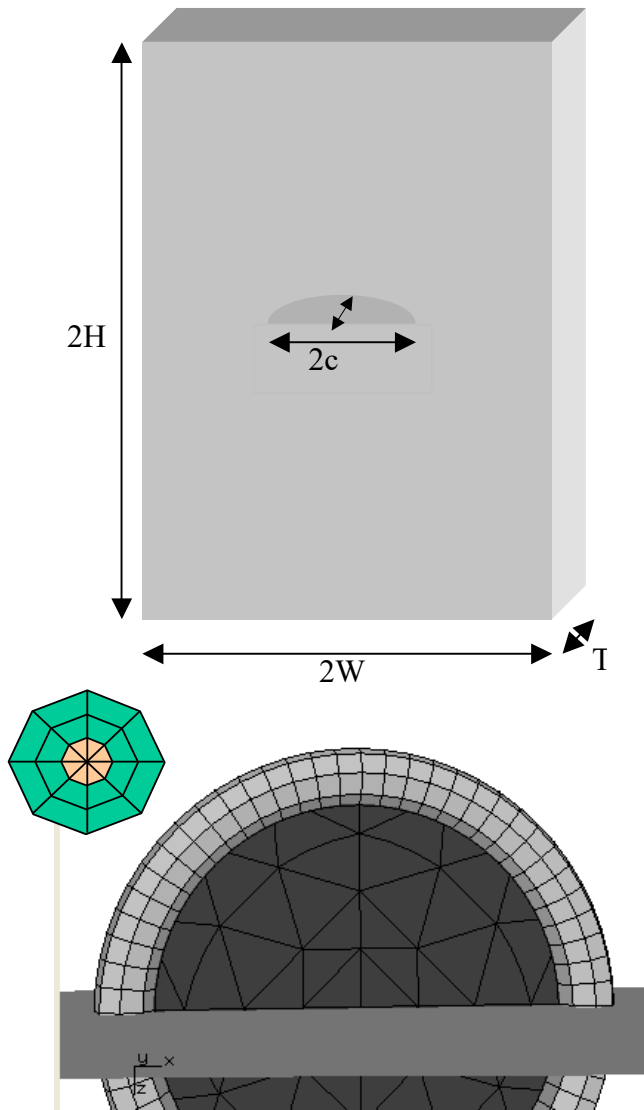


Template Cross-Section



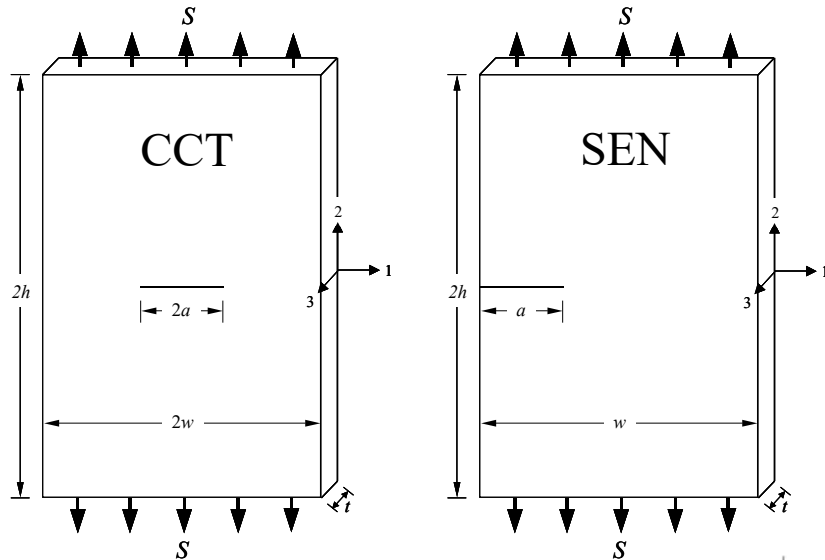
Default crack front mesh gives 0.2% error.

M-Integral SIFs: Penny Surface Crack



The Raju-Newman solution is known to be slightly inaccurate near the free surface.

Independent Mode I SIF Verification



Analyses performed by Dawn Phillips NASA Langley Research Center using FRANC3D Version 5.0

Users are strongly encouraged to do their own benchmark/validation.

Table 2. Comparison of K_I (psi $\sqrt{\text{in}}$) for CCT specimen with various values of a/w

a	a/w	$K_{I, \text{Ref.}}$	$K_{I, \text{Franc}}$	% difference
0.5 in.	0.2	1.28	1.282	-0.16
1.0 in.	0.4	1.97	1.964	0.30
1.5 in.	0.6	2.82	2.825	-0.18
2.0 in.	0.8*	4.56	4.546	0.31

* From Tada *et al.* solution only

Table 3. Comparison of K_I (psi $\sqrt{\text{in}}$) for SEN specimen with various values of a/w

a	a/w	$K_{I, \text{Ref.}}$	$K_{I, \text{Franc}}$	% difference
1.0 in.	0.2	2.43	2.420	0.41
2.0 in.	0.4	5.29	5.284	0.11
3.0 in.	0.6	12.4	12.36	0.32
4.0 in.	0.8*	42.5	42.30	0.47

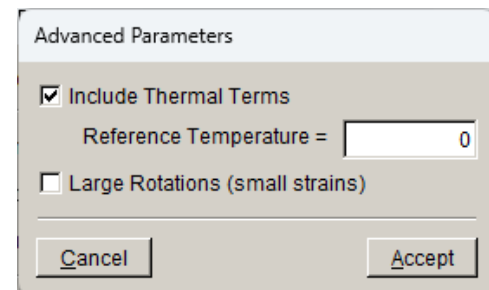
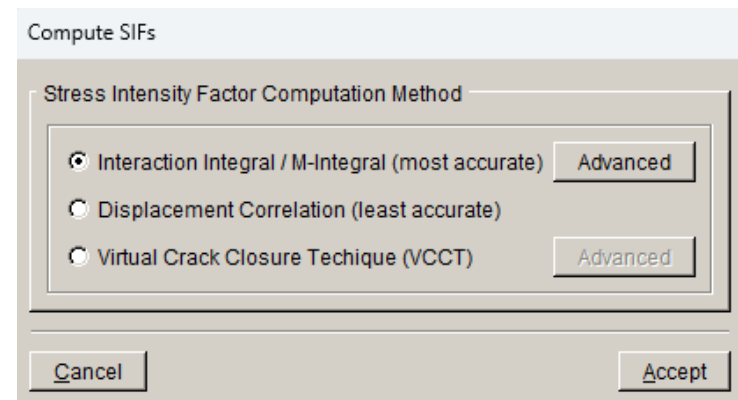
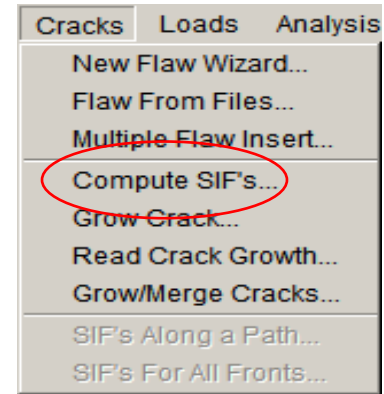
* From Tada *et al.* solution only

Computing SIFs

FRANC3D User Interface

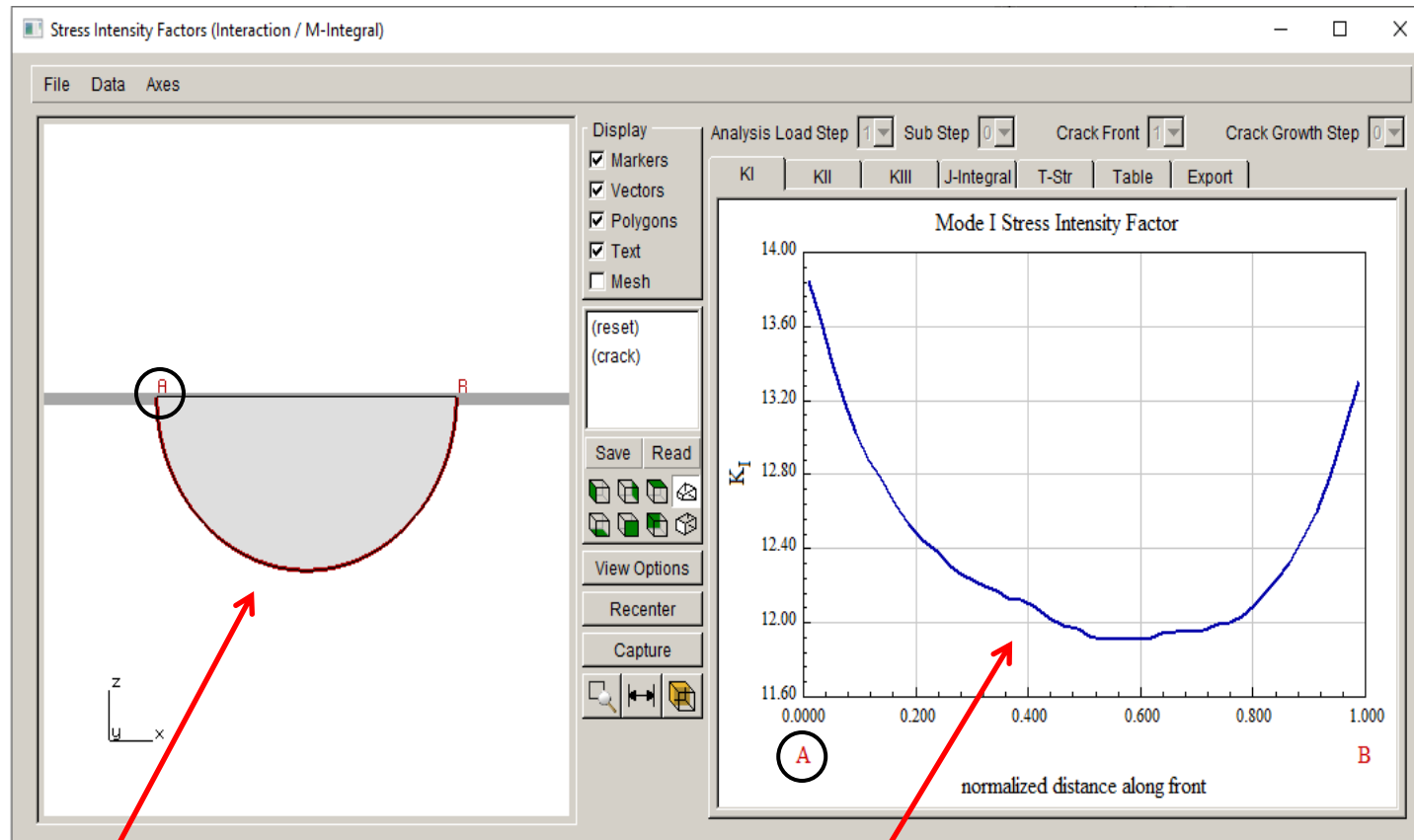
Computing SIFs User Interface

- From the Crack Menu select Compute SIFs
- Compute SIFs dialog
 - Default in FRANC3D is M-integral
 - M-integral allows the user to include thermal terms, applied crack traction, contact crack pressure, large rotation (use **Advanced** button)
 - Displacement correlation can be used to verify the M-integral extra terms are included correctly
 - Virtual Crack Closure – added for cracks in bi-material interfaces
 - If you switch SIF computation method, that method will continue to be used until you switch back!



Plotting Options

- SIF distributions are displayed in this dialog



Crack in Model

Plot the computed SIF distributions along the crack front

Plotting Options

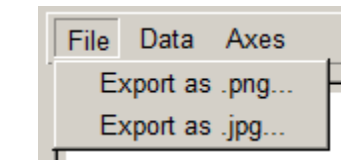
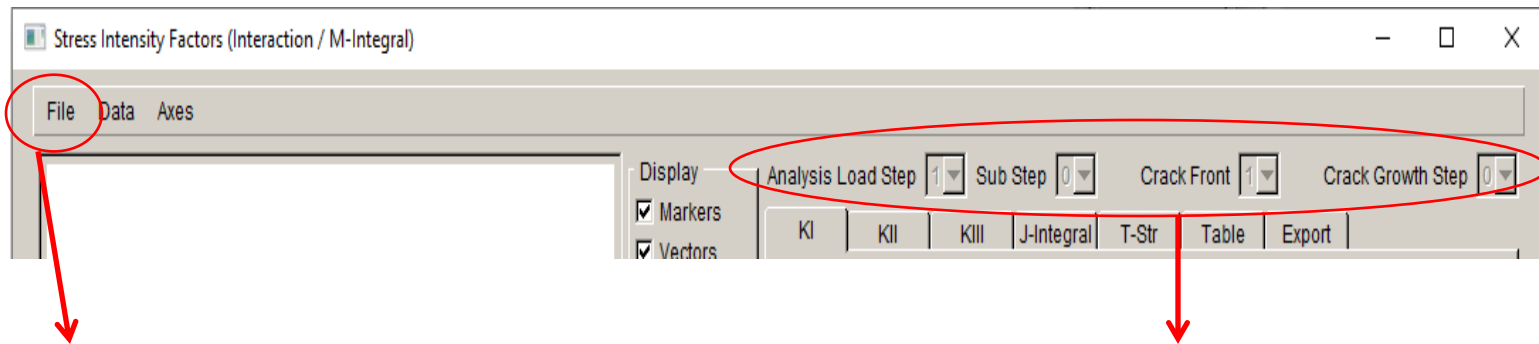
- Export tab allows the SIF data to be saved to a file

Write file button to set the file name and save the data

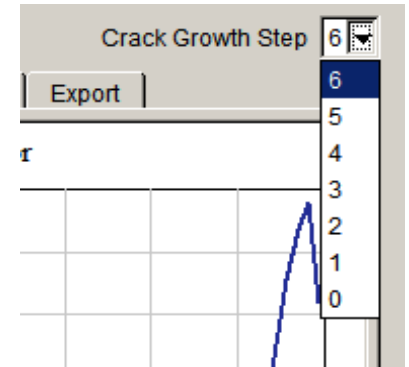
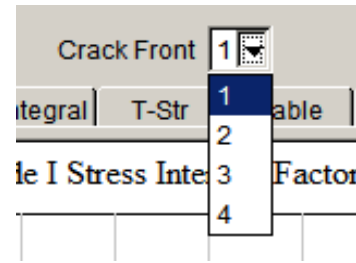
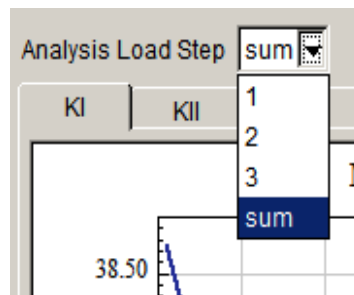
The screenshot shows a software window with the 'Export' tab selected. At the top, there are dropdown menus for 'Analysis Load Step' (set to 1), 'Sub Step' (set to FINAL), 'Crack Front' (set to 1), and 'Crack Growth Step' (set to 0). Below these are tabs for K_I , K_{II} , K_{III} , J-int, T-str, Temp, Table, and Export. The 'Export' tab is active, displaying two sections: 'Delimiters' and 'Options'. In the 'Delimiters' section, 'Spaces' is selected with a radio button, while 'Tabs' and 'Commas' are unselected. In the 'Options' section, 'Reverse order' and 'Units' are unselected. Under the 'Content' section, there are checkboxes for K_I , K_{II} , K_{III} , J-Integral, T Stress, Temperature, front coordinates, and front coordinate axes. The 'front coordinates' checkbox is checked. At the bottom of the window, there are two buttons: 'Write file' and 'Preview'. A red arrow points from the text 'Write file button to set the file name and save the data' to the 'Write file' button.

Plotting Options

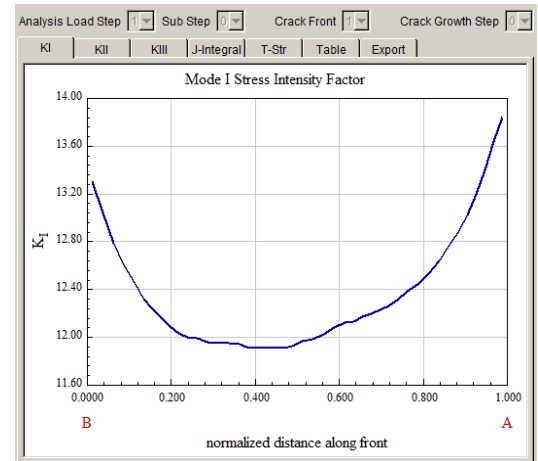
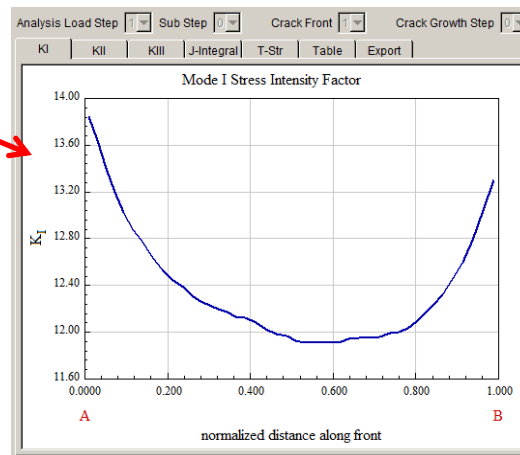
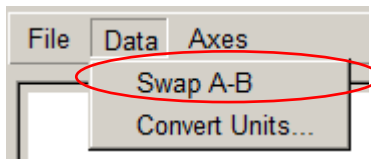
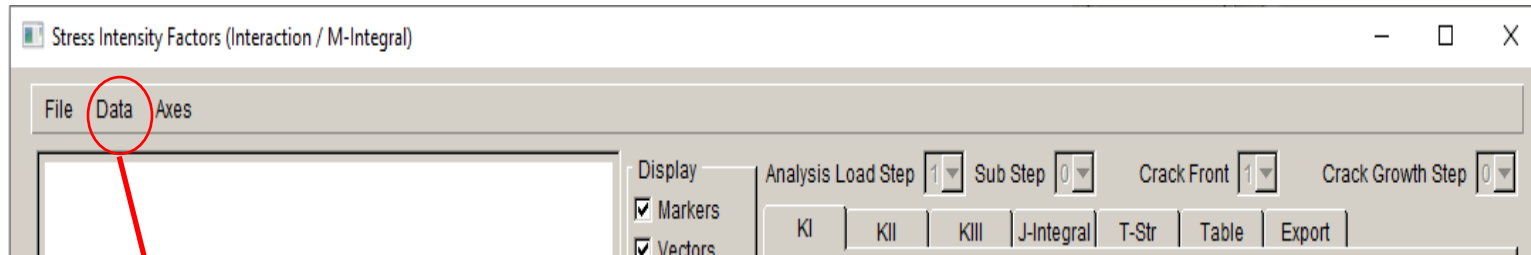
User can display SIF data from a model with multiple load steps (and substeps), multiple crack fronts, and multiple crack growth steps



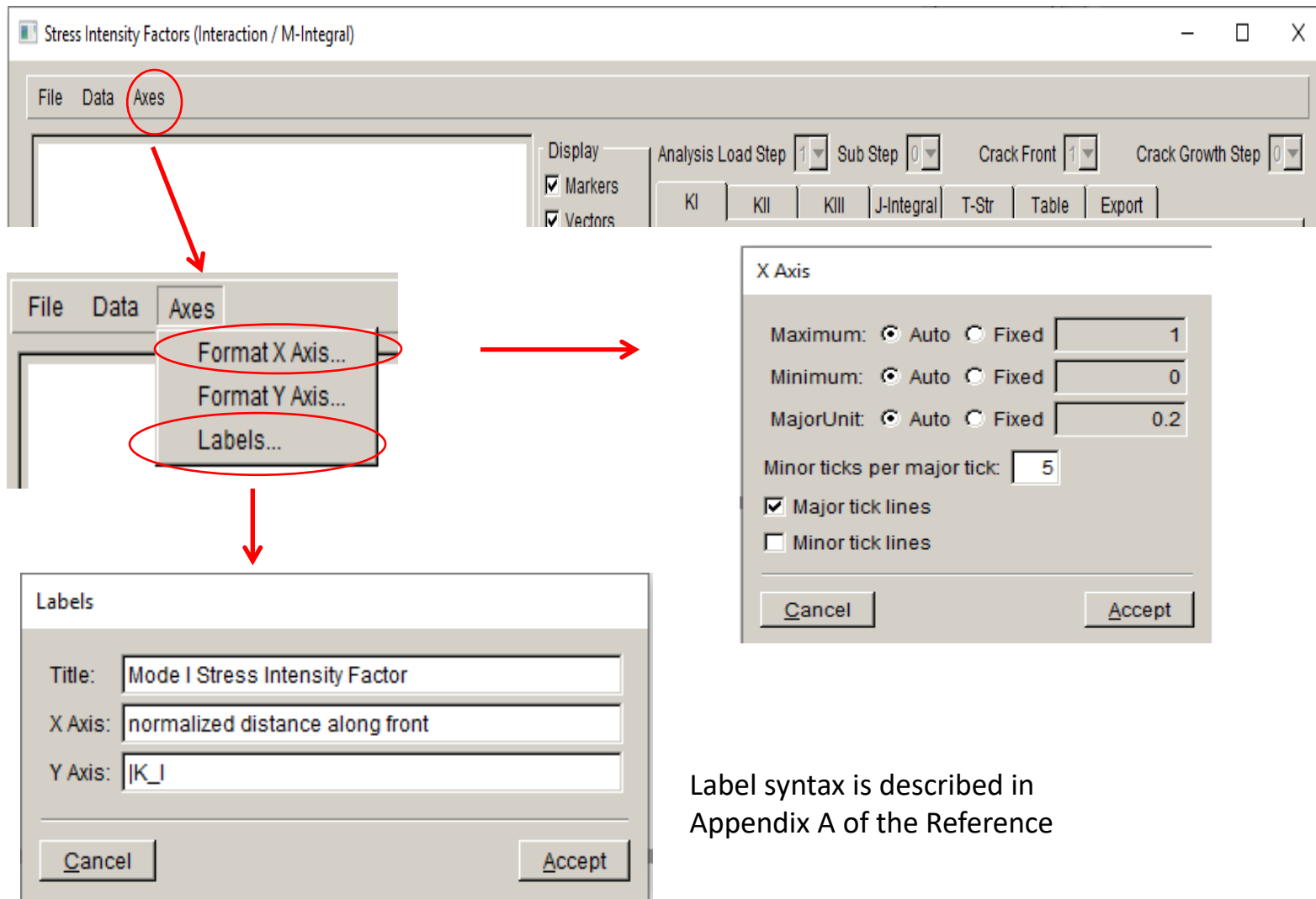
Export images of
the XY plot to a file



Plotting Options



Plotting Options



Label syntax is described in
Appendix A of the Reference

End Part 8