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1 Introduction

This manual contains some benchmark examples and provides comparisons between analytical or handbook and FRANC3D stress intensity factors (SIF). FRANC3D SIFs are computed using the M-integral approach, computed at crack front element mid-side nodes.

2 Interior Penny Shaped Crack in a Rectilinear Bar (Sneddon Solution)

An internal circular crack is simulated in the rectilinear bar shown in Figure 1. The elastic modulus is 10,000 and Poisson’s ratio is 0.0. The bar is constrained with simple supports. Unit traction is applied to left and right end surfaces as indicated by the red arrows. ANSYS is used for the initial elastic stress analysis of the uncracked model, and the resulting nodal displacements and stresses are saved to .dsp and .str files, respectively; the mesh information is saved in a .cdb file.

We compare the SIF values for applied far-field loading versus crack face pressure, as well as comparing the computed SIFs to the analytical value. We use FRANC3D’s Submodel tool to create a local portion in the middle section of the bar. The crack is centered in the model and has a radius of 0.1 units, Figure 2.

Figure 1. Rectilinear bar in ANSYS. It is 10 units long, 5 units wide and 5 units deep, with simple supports and unit uniform traction in the y-direction.
The initial crack front template radius is 0.01 units, Figure 3, with 8 elements around the crack front and 3 rings of elements (quarter-point wedge with two rings of bricks). Approximately 27,500 elements are used; the number of elements is relatively small because the local model is relatively small.

Typically one or two levels of mesh refinement are used to study the accuracy and convergence of the SIFs. The first change that we make here is to switch from FRANC3D volume meshing to ABAQUS; switching to ABAQUS meshing is a simple change when setting the crack front template parameters. The other change is to increase the number of rings of elements in the mesh template, increasing from 3 to 5; at the same time the template element aspect ratio is reduced from 2 to 1. This has the benefit of reducing the size of the elements adjacent to the crack front while keeping the pyramid and tetrahedral elements at the same distance away from the front.

The resulting Mode I SIFs are shown in Figure 4 and the % error (difference from the analytical solution) is shown in Figure 5.
The analytical solution for the Mode I SIF for this crack, in an infinite domain, was given by Sneddon (see Murakami, 1987) as:

$$ K_I = 2\sigma a \sqrt{\frac{a}{\pi}} $$

$$ K_{II} = K_{III} = 0 $$

The target value of $K_I$ is 0.357, and is constant around the crack front.

The mean value for the default FRANC3D mesh is 0.3567; the mean difference from the analytical value is 0.09%. The mean value for the default template with ABAQUS volume mesh is 0.3578; the mean difference from the analytical value is 0.22%. The mean value for the more refined template and ABAQUS volume mesh is 0.3569; the mean difference in this case is 0.02%.

All of these values are reasonably close to the analytical value. If you find that your results for this model differ by more than 0.5% from the analytical value, you should refine the volume mesh, either by adjusting the meshing parameters for FRANC3D, or using the ABAQUS volume mesh, or increasing the number of rings (and possibly the template radius also), or refining the initial uncracked mesh (and using second order elements for the uncracked mesh).
Figure 4. Mode I SIFs for the internal penny-shape crack.
2.1 Crack Face Pressure (Traction)

FRANC3D allows one to apply crack face pressure (traction). The same model and crack geometry are used here to compare the Mode I SIF solution for remote loading versus crack face traction. The only difference is that we remove the far-field traction shown in Figure 1, and instead we apply a constant crack face pressure of 1.

The refined template mesh parameters are used here; template radius is 0.01 with 5 rings of elements with aspect ratio equal to 1.

The Mode I SIFs are shown in Figure 6 and the % error (difference from reference solution) is shown in Figure 7. The average Mode I SIF is 0.3573 and the average % error is 0.08.
Figure 6. Mode I SIFs for the internal penny-shape crack with crack face pressure.

Figure 7. % error in Mode I SIFs for the internal penny-shape crack with crack face pressure.
3 Internal, Inclined (45 degrees), Penny-Shaped Crack

A 10x5x5 block is used to model an infinite body containing a centered, internal, penny-shaped crack, with radius=0.125, and inclined at 45 degrees, Figure 8. The loading is far-field uniform tension equal to 1.0 and the material properties consist of an elastic modulus of 10,000 and a Poisson’s ratio of 0.0.

The default crack front template mesh is shown in Figure 9. The default template radius is 0.0125 units with 8 rings around the crack front and 3 rings of elements (quarter-point wedge and 2 rings of bricks). Based on the results for the Sneddon crack above, we decrease the template radius to 0.01, switch the volume meshing to ABAQUS, and increase the number of rings in the template from 3 to 5. About 132,000 elements are used for the entire model.

Figure 8. Internal, inclined at 45 degrees, penny-shaped crack in a rectilinear bar.
The analytical values for Mode I, II and III SIFs for this crack configuration are available in Murakami (1987). Mode I SIF is constant at 0.200 along the crack front perimeter. The Mode II SIF is zero at the points A and B, Figure 10, and reaches a maximum (or minimum depending on the sign) at points C and D equal to 0.2 (or -0.2). Mode III SIF is zero at points C and D, and reaches a maximum at points A and B equal to 0.2. The equations for the three modes of SIF are:

\[
K_I = \sigma \sin^2 (\gamma) \sqrt{\pi r} (2/\pi)
\]
\[
K_{II} = \sigma \sin(\gamma) \cos(\gamma) \sin(\theta) \sqrt{\pi r} (2/(2 - \nu))
\]
\[
K_{III} = \sigma \sin(\gamma) \cos(\gamma) \cos(\theta) \sqrt{\pi r} (2/(2 - \nu)) (1 - \nu)
\]

where \(\gamma\) is the angle of inclination (here 45 degrees), \(\theta\) is the position along the crack front, \(r\) is the crack radius.
Figure 10. Inclined, penny-shaped crack subjected to uniform, remote tension.

Figure 11 shows all three SIF modes. The mean value of Mode I is 0.1992. The average error in Mode I SIF is 0.4%. The maximum absolute Mode II and Mode III values are 0.1993 and 0.1995.

Figure 11. Mode I, II, and III SIFs along the crack front of an internal, inclined (at 45 degrees), penny-shaped crack under unit uniform uniaxial tension.
4 Surface Ellipse Crack in a Plate – Raju-Newman Finite Element Solution

The handbook numerical solution for a surface crack in a finite plate was developed by Raju and Newman (1979; 1986); the accuracy is within 5%. The typical plate dimensions are depicted in Figure 12; the loading consists of uniform unit traction and simple constraints.

A plate was created and analyzed using ANSYS with the dimensions: \( H=4, \ W=4, \ t=2 \). The model was read into FRANC3D and a crack, with dimensions: \( a=0.8 \) and \( c=0.8 \), was inserted, Figure 13. The default crack front template mesh is shown in Figure 14. The default template radius is 0.1 units with 8 rings around the crack front and 3 rings of elements (quarter-point wedge and 2 rings of bricks). Based on the Sneddon crack results, we switch to ABAQUS volume meshing and we increase the number of rings in the template to 5. For this model, we also turn off the Do Crack Mouth Coarsen Mesh flag. Approximately 57,000 elements were used for the entire model.

To check the accuracy of the computed SIF values, a refined mesh was created by modifying the template radius and the element aspect ratio of the template elements. A radius of 0.025 units and an aspect ratio of 1 (instead of the default 2) are used. The refined mesh contained approximately 230,000 elements.

![Figure 12. Raju/Newman surface crack in a plate model configuration.](image-url)
The Mode I SIFs from FRANC3D are plotted in Figure 15 along with the Raju-Newman handbook solution. There is very little difference in Mode I SIF results for the two meshes except at the free surface. For the first mesh, the maximum difference between the FRANC3D solution and the Raju-Newman solution is at the ends of the crack front, at the plate surface; the difference is about 2%. At the mid-point, the % difference is about 0.5%. The refined template mesh captures the drop in Mode I SIF at the surface, but the rest of the solution is essentially identical.
Figure 15. Raju-Newman surface crack in a plate – Mode I SIFs.

5 Through-Thickness Crack in a Plate

A through-thickness edge crack in a plate and a middle-through-thickness crack in a plate are standard benchmark problems. The model shown in Figure 16 is a 10x5x5 plate, and the boundary conditions consist of uniform traction and simple displacement constraints. The elastic modulus is 3.0e7 and Poisson’s ratio is 0.30. The top and bottom surfaces of the plate have uniform unit traction.

Two sets of boundary conditions are applied to simulate a through-thickness edge crack and a middle-through-thickness crack. For the first case, only simple constraints are applied to prevent rigid body motion. For the second case, rollers in the x-direction are applied to the surface where the crack is inserted. A crack of 0.5 units is inserted on the left side, Figure 17.
Figure 16. Plate model.

Figure 17. FRANC3D model with an edge crack of length 0.5 units.

5.1 Through-Thickness Edge Crack

The default crack front template mesh is shown in Figure 18. The default template radius is 0.05 units with 8 rings around the crack front and 3 rings of elements (quarter-point wedge and 2 rings of bricks). Based on the earlier model results, we switch to ABAQUS
volume meshing, turn off the Do Crack Mouth Coarsen Mesh flag, and increase the number of rings to 5. These parameters are used for both the edge crack and the middle crack models. Approximately 85,000 elements are used for the crack model.

Without the roller boundary conditions on the left face, the model represents a through-thickness edge crack. The handbook, 2D solution is provided by the equation (Murakami, 1987):

\[ K_I = \sigma \sqrt{\pi a} F_I(\alpha) \]

The correction factor is:

\[ \alpha = \frac{a}{w} \]

\[ F_I(\alpha) = (1.12 - 0.231\alpha + 10.55\alpha^2 - 21.72\alpha^3 + 30.39\alpha^4) \]

for \( \frac{a}{w} \leq 0.6 \), \( K_{II} = K_{III} = 0 \)

This 2D solution is insensitive to Poisson’s ratio. In this model, \( a=0.5 \) and \( w=5 \), so the correction factor is 1.1837. Therefore, the handbook \( K_I \) value is 1.4836. The FRANC3D SIFs, which are sensitive to the value of Poisson’s ratio, are shown in Figure 19.

![Figure 18. Crack front template elements for edge crack. The left panel shows the entire crack and the right panel shows about 1/8th of the crack.](image)

The FRANC3D model clearly is neither plane strain nor plane stress; the handbook solution is a 2D solution. If we add constraints on the z-surfaces to simulate plane strain conditions, the maximum difference between the FRANC3D Mode I SIF value and the handbook value is 0.1%. The average Mode I SIF value for plane strain conditions is 1.485 compared to the handbook solution of 1.484.
5.2 Middle-Through-Thickness Crack

The handbook, 2D solution for Mode I SIF for the center crack is given in Murakami (1987). The equations are:

\[ K_I = \sigma \sqrt{\pi a} \quad F_I (\alpha, \beta) \]
\[ \alpha = \frac{2a}{w}, \quad \beta = \frac{h}{W} \]

The parameters h, W, and a are depicted in Figure 20. For this model, the correction factor, \( F_I \), is 1.014, and the value of Mode I SIF is thus 1.271. The FRANC3D Mode I SIF values are plotted in Figure 21. As for the edge crack, the FRANC3D model is neither plane strain nor plane stress. If we add constraints on the z-surfaces to simulate plane strain conditions, the average difference between the FRANC3D Mode I SIF value and the handbook value is 0.32%. The average Mode I SIF value for plane strain conditions is 1.267.

Figure 19. Mode I SIF values for a part-through edge crack. With z-constraint simulates plane strain conditions, which matches the 2D handbook solution.
Figure 20. Center cracked plate under remote tension.

Figure 21. Mode I SIF values for middle-through crack. With z-constraint simulates plane strain conditions, which matches the 2D handbook solution.

Note that the symmetry boundary conditions are applied in ABAQUS to the surface normal to the x-axis at x=0. When we read the .inp file into FRANC3D, we retain all the displacement boundary conditions without retaining the associated nodes and mesh.
facets. We do retain the nodes and mesh facets associated with the pressure boundary conditions. Because the displacement constraints are tied to nodes that are retained with the pressures, these constraints are retained as well. When we write the new .inp file for the crack model, we transfer all of the boundary conditions. This means that the symmetry boundary conditions on the surface where the crack is inserted are recreated automatically by FRANC3D. This is verified by reading the model into ABAQUS CAE and displaying the constraints, Figure 22.

![Figure 22. Boundary conditions shown in ABAQUS CAE for middle-crack model.](image)

6 Thick Plate with Middle Through Crack and Anisotropic Material Properties

A ‘thick plate’ model has been used to verify that FRANC3D produces accurate stress intensity factors using the M-Integral for isotropic and anisotropic material properties (Banks-Sills et al., 2007). Two different sets of boundary conditions are used here: 1) simple uniform tension and 2) simple shear. The outer portion is shown in Figure 23 with the mesh retained for the boundary conditions and on the cut surfaces. The inner portion is shown in Figure 24 with a crack inserted. The original plate is 30x30x15 units and the through crack is 2 units wide; plane strain conditions are approximated along the crack front in the middle of the plate. The plate is first analyzed with uniform tension and simple supports, Figure 25. The material properties are provided in Table 1.
Figure 23. Outer portion of the thick plate model with tension boundary conditions.

Table 1: Material Properties

<table>
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<tr>
<th></th>
<th>$E_x$, $E_y$, $E_z$</th>
<th>$N_{uxy}$, $N_{uyz}$, $N_{uzx}$</th>
<th>$G_{xy}$, $G_{yz}$, $G_{xz}$</th>
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<td>0.3, 0.3, 0.3</td>
<td></td>
</tr>
<tr>
<td>orthotropic</td>
<td>950, 950, 2400</td>
<td>0.45, 0.3, 0.3</td>
<td>328, 231, 231</td>
</tr>
</tbody>
</table>
Figure 24. Thick plate inner portion with a middle through crack.

Figure 25. Thick plate with a middle through crack and tension boundary conditions and simple constraints.
6.1 Uniform Tension

The stress analyses were performed using ANSYS. The deformed shape, with maximum principal stress contours, is shown in Figure 26.

![Figure 26. Thick plate deformed crack with maximum principal stress contours.](image)

The analytical solution for the Mode I SIFs for the isotropic case under uniform tension is defined as: \( K_I = \sigma \sqrt{a} \) where \( \sigma=1 \) and \( a=1 \), resulting in \( K_I=1.772 \). The SIFs are plotted in Figure 27.
6.2 Pure Shear

The second set of boundary conditions represents pure shear; Figure 28 shows the deformed shape from ANSYS of the original uncracked plate. The SIFs for pure shear are plotted in Figure 29. Note that the application of pure shear in ANSYS requires the use of surface effect elements and the shear is applied in the element coordinate system. As for the tension case, the outer portion of the model was retained as a global model as seen in Figure 23, and the inner portion was extracted for crack insertion and remeshing.
Figure 28. Deformed shape under pure shear boundary conditions.

Figure 29. SIFs for pure shear for isotropic and anisotropic material properties.
7 Corner Crack in a Plate with a Hole

A plate with a corner cracked hole is another standard benchmark problem. The uncracked plate is created using ANSYS and is 40x40x1 units with a hole of radius 1 unit, Figure 30; note that only half of the plate is modeled and symmetry boundary conditions are used, which implies that there will be two symmetric cracks emanating from the hole (this is consistent with the handbook solution). The elastic modulus is set to 10,000 and Poisson’s ratio is 0.3. The boundary conditions consist of uniform tension and simple displacement constraints.

A quarter-circular corner crack is inserted at the edge of the hole; the radius of the crack is 0.05 units. The default crack front template mesh is shown in Figure 31; the template radius is 0.005 units. Based on previous model results, we switch to ABAQUS volume
meshing, turn off the Do Coarsen Crack Mouth Mesh flag and increase the number of rings in the template to 5.

Approximately 32,000 elements are used for entire model. The subsequent analysis is done using ANSYS; the deformed shape and the maximum principal stress contours near the corner crack are shown in Figure 32.

Figure 31. Plate with hole and corner crack - front template mesh and final surface mesh. The entire circular crack and template is shown in the left panel; the program determines the intersections and discards the portion outside the model.
Figure 32. Plate with hole and corner crack - deformed shape and maximum principal stress contours.

The stress is higher on the inside of the hole than on the surface of the plate, and the Mode I SIF reflects this, Figure 33. The handbook numerical solution is available in Murakami (1987) and from Newman and Raju (1986). The equations from Newman and Raju (1986) have been encoded in Excel and the handbook values are plotted in Figure 33 also. The difference between the computed and handbook values is significant (~10%). Additional reference solutions (Shin, 1990; Lin and Smith, 1999) indicate similar discrepancies with the Newman and Raju (1986) solution for this model.

The Mode I SIF curves for Mu=0.0 and for Mu=0.45 are shown in Figure 34. For Mu=0.0, most of the curve matches the handbook solution, although the FRANC3D values at both ends are significantly higher than the handbook values.
Figure 33. Mode I SIF for corner crack in plate with hole. Position 0 is on the surface inside the hole and position 1 is on the front plate surface.
Figure 34. Mode I SIF for corner crack in plate with hole with various Poisson ratios.
References


